Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2019

Take-Home Final Examination
Due two weeks from receipt,
or 18 December, whichever is earlier.

Instructions: Do both of parts **Set** and **Game**, and, if you wish, part **Match** as well. *Show all your work.* You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *otherwise you may not consult any sources, nor consult or work with any other person on this exam.*

Part Set. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Recall that the inhabitants of the Island of Knights and Knaves are either knights, who always tell the truth, or knaves, who always lie. While visiting the Island you encounter nine inhabitants: Mel, Bob, Sue, Bill, Alice, Zippy, Ted, Abe and Rex.

Mel tells you that only a knave would say that Alice is a knave. Bob claims that neither Alice nor Rex are knights. Sue tells you that Alice could claim that Bill is a knight. Bill claims, "Mel could say that Zippy is a knave." Alice tells you, "I know that Rex is a knight and that Abe is a knave." Zippy says that either Bob is a knight or Ted is a knave. Ted claims that neither Bill nor Zippy are knights. Abe tells you that Sue is a knave. Rex says that Alice is a knight or Ted is a knave.

Determine, as completely as you can, whether each of the nine is a knight or a knave.

2. The Fibonacci numbers, f_n for each $n \in \mathbb{N}$, are given by the following recursive definition: $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{N}$. Use induction to show that $f_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5} \cdot 2^n}$ for all $n \in \mathbb{N}$.

3. Informally, $\mathbb{H}$ is the set of the *quaternions*, which are numbers of the form $a + bi + cj + dk$, where a, b, c, $d \in \mathbb{R}$ and i, j, and k satisfy the following equations: $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, and $ik = -j$. (Addition and multiplication work as in the reals otherwise.) Give a formal definition of the quaternions, as well as of $+$ and $\cdot$ in terms of your formal definition.

4. Suppose a and b are schnitts. Determine whether $D = \{\, s - t \mid s \in a \wedge t \in b \,\}$ is a schnitt. If it isn't, check to see which parts of the definition of schnitt D does satisfy.

Part Game. Do any *four* (4) of problems **5** – **10**. *[40 = 4 × 10 each]*

5. Show that every natural number n is equal to a sum of the form

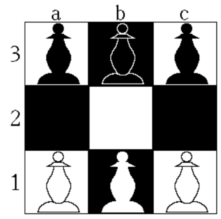
$$a_k \cdot k! + a_{k-1} \cdot (k-1)! + \cdots + a_2 \cdot 2! + a_1 \cdot 1! + a_0 \cdot 0!$$

for some $k \geq 0$ and such that each a_i is a natural number with $0 \leq a_i \leq i$.

More exam questions on page 2 . . .

... and here they are:

6. The game of *hexapawn* is played on a 3×3 chess board with three pawns on each side, initially set up as in the diagram below.



The players, White and Black, take turns moving pawns. Each may move any one pawn of their own colour on their move. Each pawn may be moved in two different ways: it may be moved one square forward, or it may capture a pawn of the other colour one square diagonally ahead of it. A pawn may not be moved forward if there is a pawn in the next square. A player loses if they have no legal moves available on their turn or if the other player reaches the end of the board with a pawn, and wins if the other player loses.

a. Show that hexapawn is a finite game, *i.e.* a game that cannot go on forever. *[5]*

b. Find a winning strategy for one player or the other. *[5]*

7. Suppose $\{a_n \mid n \in \mathbb{N}\}$ is a strictly increasing sequence of real numbers, *i.e.* $n < m$ implies that $a_n < a_m$, with $a = \sup\{\, a_n \mid n \in \mathbb{N} \,\}$. Show that for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n - a| < \varepsilon$.

8. A natural number $n > 1$ is said to be *perfect* if it is the sum of its divisors other than itself. (For example, $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$ are the two smallest perfect numbers.) Suppose p and $2^p - 1$ are both prime. Show that $n = 2^{p-1}\left(2^p - 1\right)$ is perfect.

9. *Pentominoes* are shapes obtained by gluing five 1×1 squares together full edge to full edge. Two pentominoes that can be made congruent via reflections (*i.e.* flips) or rotations are considered to be the same. Find all twelve pentominoes and an arrangement of all of them into a 5×12 rectangle.

10. Suppose that instead of using schnitts, the real numbers were defined as equivalence classes of Cauchy sequences of rational numbers:

> A sequence $\{q_n\}$ is *Cauchy* if for every $\epsilon > 0$, there is an N such that for all $m, k \geq N$, $|q_m - q_k| < \varepsilon$. Define an equivalence relation $\approx$ on Cauchy sequences of rationals by $\{q_n\} \approx \{p_n\}$ if for all $\varepsilon > 0$, there is an N such that for all $n \geq N$, $|q_n - p_n| < \varepsilon$. Then $\mathbb{R} = \left\{\, [\{q_n\}]_\approx \mid \{q_n\} \text{ is a Cauchy sequence of rationals} \,\right\}$.

Define multiplication on the real numbers using this definition and check that it is commutative.

[Total = 80]

Part Match. Bonus time!

$\boldsymbol{\xi}$. Write an original poem about logic or mathematics. *[1]*

I HOPE THAT YOU HAD A GOOD COURSE. HAVE A BETTER BREAK!