Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2019

Assignment the Extra
Not Not Nothing
Due on Wednesday, 4 December.

Here is a formal definition of a minimal language for propositional logic:

The *symbols* of the language are as follows:

Atomic formulas: A_0, A_1, A_2, ...
Connectives: $\neg$, $\rightarrow$
Parentheses: (,)

Just to be paranoid: all of the above symbols are distinct, none is a substring of any other, and there are no other symbols in the language.

The *formulas* of the language are defined as follows:

1. Every atomic formula A_i is a formula of the language.
2. If φ and ψ are any formulas of the language, then $(\neg\varphi)$ and $(\varphi \rightarrow \psi)$ are also formulas of the language.
3. No string of symbols of the language is a formula of the language unless it was formed using finitely many applications of rules 1 and 2 above.

The *logical axiom schema* for this variant of propositional logic are as follows. If α, β, and γ are any formulas of the language, then each of the following is an axiom, which may be assumed to be true without further ado:

A1. $(\alpha \rightarrow (\beta \rightarrow \alpha))$
A2. $((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)))$
A3. $(((\neg\beta) \rightarrow (\neg\alpha)) \rightarrow (((\neg\beta) \rightarrow \alpha) \rightarrow \beta))$

The only *rule of procedure* in this variant of propositional logic is *Modus Ponens*. Suppose δ and η are any formula of the language.

MP. If δ and $(\delta \rightarrow \eta)$ are both true, then η is true.

Finally, a *deduction* from some set of hypotheses H in this variant of propositional logic is a finite sequence of formulas, say $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n$ for some $n \geq 1$, such that each φ_k is in H, is an instance of one of the logical axiom schema, or follows from preceding formulas in the deduction by Modus Ponens, *i.e.* there are φ_i and φ_j for some i, $j < k$ earlier in the sequence such that φ_j is $(\varphi_i \rightarrow \varphi_k)$.

For example, here is a deduction from an empty set of hypotheses, *i.e.* using only the logical axiom schema and Modus Ponens, of $(\varphi \rightarrow \varphi)$, where φ could be any formula of the language:

1. $(\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi))$ [A1]
2. $((\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow ((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)))$ [A2]
3. $((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi))$ [1,2 MP]
4. $(\varphi \rightarrow (\varphi \rightarrow \varphi))$ [A1]
5. $(\varphi \rightarrow \varphi)$ [3, 4 MP]

1. Give a deduction of $((\neg(\neg\varphi)) \rightarrow \varphi)$ from an empty set of hypotheses. *[5]*

2. Give a deduction of $(\varphi \rightarrow (\neg(\neg\varphi)))$ from an empty set of hypotheses. *[5]*