Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2019

Assignment #9
Real Bounds
Due on Monday, 11 November.

Recall that a *schnitt* or *Dedekind cut* is a subset S of $\mathbb{Q}$ such that:

1. $S \neq \emptyset$ and $S \neq \mathbb{Q}$.
2. S is "closed downwards": if $q \in S$ and $p \in \mathbb{Q}$ with $p < q$, then $p \in S$.
3. S has no largest element: if $q \in S$, then there is an $r \in S$ with $q < r$.

Intuitively, a schnitt S is $(-\infty, s) \cap \mathbb{Q}$ for some real number s. Formally, the schnitt S *is* the real number s. That is, we define the set of real numbers to be $\mathbb{R} = \{ S \subset \mathbb{Q} \mid S \text{ is a schnitt} \}$. With this definition, it is pretty easy to define certain common real numbers as schnitts, *e.g.* $0_{\mathbb{R}} = \{ q \in \mathbb{Q} \mid q < 0 \}$ and $1_{\mathbb{R}} = \{ q \in \mathbb{Q} \mid q < 1 \}$, the operation of addition by $S + T = \{ a + b \mid a \in S \text{ and } b \in T \}$, and the linear order on the reals by $S \leq T \iff S \subseteq T$. (Things get more complicated when defining multiplication, unfortunately.)

This assignment is dedicated to showing that the linear order on the real numbers is *complete*: that every non-empty set of real numbers with an upper (respectively, lower) bound has a least upper (respectively, greatest lower) bound. Here are the relevant definitions:

- If $\emptyset \neq A \subset \mathbb{R}$, then $u \in \mathbb{R}$ is an *upper bound* for A if $a \leq u$ for all $a \in A$.
- If $\emptyset \neq A \subset \mathbb{R}$, then $v \in \mathbb{R}$ is a *lower bound* for A if $v \leq a$ for all $a \in A$.
- If $\emptyset \neq A \subset \mathbb{R}$ has an upper bound, then the *supremum* or *least upper bound* of A is the real number $\sup(S)$ such that $\sup(S)$ is an upper bound for A and $\sup(S) \leq u$ for every upper bound u of A.
- If $\emptyset \neq A \subset \mathbb{R}$ has a lower bound, then the *infimum* or *greatest lower bound* of A is the real number $\inf(S)$ such that $\inf(S)$ is a lower bound for A and $v \leq \inf(S)$ for every lower bound v of A.

In what follows, you may assume that the usual algebraic operations and the linear order on the rationals have all the usual properties. (Do note that the linear order on the rationals is *not* complete.)

1. Show that every non-empty set $A \subset \mathbb{R}$ with an upper bound has a supremum. *[5]*

2. Show that every non-empty set $A \subset \mathbb{R}$ with a lower bound has an infimum. *[5]*