Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2019

Assignment #5 Make-Up
The Natural Numbers as a Well-Order
Due sometime or other - don't rush!.

This assignment is a make-up for Assignment #5, on which a number of students missed at least part of the point of the question. If you do this assignment, and do better than you did on Assignment #5, this one will count instead of Assignment #5 in computing your mark.

Recall from class that we defined the natural numbers from the empty set $\emptyset$ and the successor function $S(x) = x \cup \{x\}$ as follows: $0 = \emptyset$, $1 = S(0) = 0 \cup \{0\} = \{0\}$, $2 = S(1) = 1 \cup \{1\} = \{0, \{0\}\}$, $3 = S(2) = 2 \cup \{2\} = \{0, \{0\}, \{0, \{0\}\}\}$, and so on. In general, the immediate successor of the natural number n is $S(n) = n \cup \{n\}$. The set of all natural numbers, guaranteed to actually exist by the axiom of infinity, is usually called $\mathbb{N}$.

One of the advantages of this definition of the natural numbers is that is very easy to define the usual linear order $<$, namely by $n < k$ if and only if $n \in k$. In Assignment #5 the two problems were to show that $\in$ on $\mathbb{N}$ has two of the basic properties of a linear order, transitivity and trichotomy. Your task in this assignment is to show that it has an additional property, namely the well-ordering property. This property is closely connected to the fact that we can use the natural numbers as a framework for induction.

1. Show that *in* has the well-ordering property on $\mathbb{N}$: if A is any non-empty subset of $\mathbb{N}$, then A has a least element in the linear order given by $\in$ on $\mathbb{N}$. *[5]*

2. Explain the connection between the well-ordering property and induction. *[5]*

NOTE: It may be helpful here or there to use a more sophisticated version of the Axiom of Foundation: *If x is a non-empty set, then there is an element $y \in x$ such that $y \cap x = \emptyset$.* Notice that this disallows having any set b such that $b \in b$: if such a set b existed, then $\{b\}$ would violate the Axiom of Foundation.