Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2019

Assignment #5
The Natural Numbers as a Linear Order
Due on Friday, 11 October.

Recall from class that we defined the natural numbers from the empty set $\emptyset$ and the successor function $S(x) = x \cup \{x\}$ as follows: $0 = \emptyset$, $1 = S(0) = 0 \cup \{0\} = \{0\}$, $2 = S(1) = 1 \cup \{1\} = \{0, \{0\}\}$, $3 = S(2) = 2 \cup \{2\} = \{0, \{0\}, \{0, \{0\}\}\}$, and so on. In general, the immediate successor of the natural number n is $S(n) = n \cup \{n\}$. The set of all natural numbers, guaranteed to actually exist by the axiom of infinity, is usually called (N).

One of the advantages of this definition of the natural numbers is that is very easy to define the usual linear order $<$, namely $n < k$ if and only if $n \in k$. This assignment is all about showing that this way of defining $<$ works.

1. Show that if n, k, $m \in \mathbb{N}$ and $n \in k$ and $k \in m$, then $n \in m$. *5]*

2. Show that if n, $k \in \mathbb{N}$, then exactly one of $n \in k$, $k \in n$, or $n = k$ is true. *[5]*

NOTE: It may be helpful here or there to use a more sophisticated version of the Axiom of Foundation: *If x is a non-empty set, then there is an element $y \in x$ such that $y \cap x = \emptyset$.* Notice that this disallows having any set b such that $b \in b$: if such a set b existed, then $\{b\}$ would violate the Axiom of Foundation.