Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2017

Solutions to Assignment #9
Some counting

1. Show that if $n \geq 1$, then $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$. *[2]*

Solution. Here we go:

$$\begin{aligned} 0 = 0^n = (1+(-1))^n &= \sum_{k=0}^{n} \binom{n}{k} 1^k (-1)^{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \\ &= (-1)^0 \binom{n}{0} + (-1)^1 \binom{n}{1} + (-1)^2 \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} \qquad \blacksquare \end{aligned}$$

2. What does $\displaystyle\sum_{k=0}^{n} \frac{1}{k!(n-k)!}$ add up to? Why? *[2]*

Solution. Observe that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} = n! \sum_{k=0}^{n} \frac{1}{k!(n-k)!}$$

It follows that $\displaystyle\sum_{k=0}^{n} \frac{1}{k!(n-k)!} = \frac{2^n}{n!}$. $\blacksquare$

3. Without using the fact that $e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$, show that $\displaystyle\left(\sum_{k=0}^{\infty} \frac{1}{k!}\right)^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!}$. *[2]*

Solution. $\displaystyle\left(\sum_{k=0}^{\infty} \frac{1}{k!}\right)^2 = \left(\sum_{k=0}^{\infty} \frac{1}{k!}\right)\left(\sum_{m=0}^{\infty} \frac{1}{m!}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{\infty} \frac{1}{m!}\right) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{m!}$, and the trick now is to rearrange this sum of products and apply the result of question **2** above:

$$\begin{aligned} \left(\sum_{k=0}^{\infty} \frac{1}{k!}\right)^2 &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{m!} = \sum_{n=0}^{\infty} \left[\sum_{k+m=n} \frac{1}{k!} \cdot \frac{1}{m!}\right] = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{1}{(n-k)!}\right] \\ &= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{1}{k!(n-k)!}\right] = \sum_{n=0}^{\infty} \frac{2^n}{n!} \qquad \blacksquare \end{aligned}$$

4. Let ${}^{\mathbb{N}}2 = \{\, f \mid f : \mathbb{N} \to \{\, 0,\, 1 \,\} \,\}$ be the set of all functions from $\mathbb{N}$ to $2 = \{\, 0,\, 1 \,\}$, and let $\mathcal{P}(\mathbb{N}) = \{\, X \mid X \subseteq \mathbb{N} \,\}$ be the set of all subsets of $\mathbb{N}$. Show that $\left|{}^{\mathbb{N}}2\right| = |\mathcal{P}(\mathbb{N})|$. *[4]*

Solution. Define a function $\chi : {}^{\mathbb{N}}2 \to \mathcal{P}(\mathbb{N})$ by setting $\chi(f) = \{\, n \in \mathbb{N} \mid f(n) = 1 \,\}$. We will show that χ is both 1–1 and onto.

First, suppose $f,\, g \in {}^{\mathbb{N}}2$ and $f \neq g$. This means that there is some $k \in \mathbb{N}$ such that $f(k) \neq g(k)$, so either $f(k) = 1$ and $g(k) = 0$, or $f(k) = 0$ and $g(k) = 1$. In the former case, $k \in \chi(f)$ but not in $\chi(g)$, and in the latter case, $k \in \chi(g)$ but not in $\chi(f)$. Either way, $\chi(f) \neq \chi(g)$, and so χ must be 1–1.

Second, suppose $A \subset \mathbb{N}$. Define $f \in {}^{\mathbb{N}}2$ by $f(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases}$. It then follows from the definition of χ that $\chi(f) = \{\, n \in \mathbb{N} \mid f(n) = 1 \,\} = A$. Thus χ must also be onto.

Since $\chi : {}^{\mathbb{N}}2 \to \mathcal{P}(\mathbb{N})$ is both 1–1 and onto, it follows by the definition of $|X|$ that $\left|{}^{\mathbb{N}}2\right| = |\mathcal{P}(\mathbb{N})|$. ■