Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2017

Solutions to Assignment #8
The unkindest cut of all?

We defined the set of real numbers in class using equivalence classes of Cauchy sequences of rational numbers. This definition makes it fairly easy to define the basic arithmetic operations, at the cost of being tedious – though pretty easy – to check that definitions work and have the usual algebraic properties. It's a bit harder to define $<$ on the real numbers and show it has the usual prperties using this approach, though. The main alternate method for defining the real numbers, using *schnitts* or *Dedekind cuts*, makes it fairly easy to define $<$ and establish its properties, but at the cost of making the definition of the arithmetic operations (and obtaining their basic properties) somewhat more cumbersome.

DEFINITION. A *schnitt* or *Dedekind cut* is a subset $S \subseteq$ of the rational numbers satisfying the following conditions:

i. $S \neq \emptyset$ and $S \neq Q$.
ii. S has no greatest element, *i.e.* if $p \in S$, then there is a $q \in S$ such that $p < q$.
iii. S is closed downward, *i.e.* if $p \in S$ and $r \in \mathbb{Q}$ with $r < p$, then $r \in S$.

Using schnitts, the set of real numbers is simply the collection of all schnitts, *i.e.* $\mathbb{R} = \{ S \mid S \text{ is a schnitt} \}$. The linear order $<$ on the real numbers is then defined by $S < T$ if and only if $S \subsetneq T$.

Intuitively, each real number r corresponds to the schnitt $R = \{ p \in \mathbb{Q} \mid p < r \}$.

1. Show that $<$, as defined above, is a strict linear order on $\mathbb{R}$. *[5]*

SOLUTION. We need to show that $<$ is irreflexive and transitive, and also satisfies trichotomy.

First, for any schnitt S we have $S = S$, so it cannot be the case that $S \subsetneq S$, and so $S \not< S$. Hence $<$ is irreflexive.

Second, suppose $R < S$ and $S < T$ for some schnitts R, S, and T. By definition, this means that $R \subsetneq S$ and $S \subsetneq T$, from which it follows – as it would for any sets – that $R \subsetneq T$, and hence that $R < T$. Thus $<$ is transitive.

Third, suppose S and T are two schnitts. We need to show that exactly one of $S = T$, $S < T$, or $T < S$ is true. If $S = T$, then neither $S < T$ nor $T < S$ hold because $<$ is irreflexive. Suppose, on the other hand, that $S \neq T$. We need to show that exactly one of $S < T$ or $T < S$ must be true in this case. If $S \neq T$ as sets, there must be some $s \in S$ such that $s \notin T$ or some $t \in T$ such that $t \notin S$. Suppose that the former is true. If $s \in S$ but $s \notin T$, then $t < s$ for every $t \in T$ because if $s \leq t$ for some $t \in T$, condition *iii* of the definition of a schnitt would require $s \in T$. Since $t < s$ for every $t \in T$ in this case and $s \in S$, condition *iii* ensures that $t \in S$, so $T \subsetneq S$, *i.e.* $T < S$. A similar argument shows that $S < T$ if there is a $t \in T$ such that $t \notin S$. Thus if $S \neq T$, one of $S < T$ or $T < S$

must hold. Note also that since $S \subsetneq T$ and $T \subseteq S$ can't both be true [Why?], only one of $S < T$ or $T < S$ is true if $S \neq T$. Thus $<$ satisfies trichotomy. ■

Recall from somewhen before calculus that a set X of real numbers has an upper (respectively, lower) bound if there is a real number which is greater (respectively, less) than or equal to every real number in X. A least upper bound for X is an upper bound for X that is less than or equal to every upper bound for X. (Similarly, a greatest lower bound for X is ...)

2. Suppose that a set $X \neq \emptyset$ of real numbers (using the schnitt definition above) has an upper bound. Show that X has a least upper bound. *[5]*

SOLUTION. Suppose V is an upper bound for X, *i.e.* $S \leq V$ for every $S \in X$. Let $U = \bigcup X = \bigcup_{S \in X} S = \{ q \in \mathbb{Q} \mid \exists S \in X : q \in S \}$. We claim that U is the least upper bound of X.

First, we need to verify that U is itself a schnitt, since otherwise it isn't even a real number as these are defined for this assignment:

> Note that since each $S \in X$ is a subset of $\mathbb{Q}$, U must also be a subset of $\mathbb{Q}$.
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> Since $X \neq \emptyset$ there is at least one schnitt in X which is itself nonempty by condition i of the definition of a schnitt; it follows that $U = \bigcup X = \bigcup_{S \in X} S$ is also not empty. The upper bound V is a schnitt, so $V \neq \mathbb{Q}$ by condition i. Since $S \leq V$ – *i.e.* $S \subseteq V$ – for every S in X, it follows that $U = \bigcup X = \bigcup_{S \in X} S \subseteq V$, and because $V \neq \mathbb{Q}$ it follows that $U \neq \mathbb{Q}$ as well. Thus U satisfies condition i.
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> Suppose that $p \in U$. Since $U = \bigcup X = \bigcup_{S \in X} S$, there is some schnitt $S \in X$ with $p \in S$. Since S is a schnitt, there is some $q \in S$ with $p < q$ by condition ii. But then $q \in U = \bigcup X = \bigcup_{S \in X} S$ as well. Thus U also satisfies condition ii for being.
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> Suppose that $p \in U$. Since $U = \bigcup X = \bigcup_{S \in X} S$, there is some schnitt $S \in X$ with $p \in S$. Since S is a schnitt, $r \in S$ for every rational number $r < p$ by condition iii. But then every rational number $r < p$ is also in $U = \bigcup X = \bigcup_{S \in X} S$, so U also satisfies condition iii.
>
> Since it satisfies all the conditions of the definition of a schnitt, U must indeed be a schnitt.

It is pretty trivial to show that U is an upper bound for X: if $S \in X$, then $S \subset \bigcup_{S \in X} S = U$, so $S \leq U$. U is the least upper bound of X if $U \leq W$ for every upper bound W of X. Suppose that W is any upper bound for X. Then $S < W$, *i.e.* $S \subseteq W$, for every $S \in X$, and so $U = \bigcup_{S \in X} S \subseteq W$, *i.e.* $U \leq W$, as required. ■