Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2017

Assignment #6
The integers

Recall from class that we defined the natural numbers from the empty set, $\emptyset$, and the *successor function*, $S(x) = x \cup \{x\}$, and then proceeded to define addition of natural numbers by recursion from the successor function: $n + 0 = n$ and, given that $n + k$ has been defined, $n + S(k) = S(n + k)$. Multiplication of natural numbers was then defined by recursion from addition in a similar way. We also defined the *predecessor function*, $P(0) = 0$ and $P(S(k)) = k$, and used it to define a difference function, $D(a, b) = a - b$ if $a > b$ and $D(a, b) = 0$ otherwise, which is as close as you can get to proper subtraction without having negative numbers. This assignment is concerned with building the set of integers, $\mathbb{Z}$, from the natural numbers using equivalence relations.

Definition. Let $\mathbb{N} \times \mathbb{N} = \{ (a, b) \mid a,\, b \in \mathbb{N} \}$ be the collection of all ordered pairs of natural numbers. Define a binary relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ by letting $(a, b) \sim (c, d)$ if and only if $a + d = c + b$.

Informally, $(a, b) \sim (c, d)$ exactly when $a - b = c - d$. (This has to be informal at the moment since we don't have real subtraction yet because the natuiral numbers don't include negatives.)

1. Show that $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$. *[4]*

Solution. We need to show that $\sim$ is reflexive, symmetric, and transitive.

$\sim$ *is reflexive:* By definition, $(a, b) \sim (a, b) \iff a + b = a + b$, which last is always true.

$\sim$ *is symmetric:* By the definition of $\sim$ and the fact that $=$ is symmetric, $(a, b) \sim (c, d) \iff a + d = c + b \iff c + b = a + d \iff (c, d) \sim (a, b)$, as required.

$\sim$ *is transitive:* Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. By definition, this means that $a + d = c + b$ and $c + f = e + d$, so $(a + d) + (c + f) = (b + c) + (e + d)$. Since addition on $\mathbb{N}$ is associative and commutative, we can rearrange and regroup on both sides of the last equation to get $(a+f)+(c+d) = (e+b)+(c+d)$. Then $a+f = D((a + f) + (c + d), c + d) = D((e + b) + (c + d), c + d) = e + b$ (alternatively, use the fact that addition on $\mathbb{N}$ satisfies the cancellation law to cancel the $c + d$ terms), from which it follows that $(a, b) \sim (e, f)$, as required.

Since the binary relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ is reflexive, symmetric, and transitive, it is an equivalence relation. ■

Definition. Denote the equivalence class of $(a, b) \in \mathbb{N} \times \mathbb{N}$ by $[(a, b)]_\sim$. Then $\mathbb{Z} = \{ [(a, b)]_\sim \mid (a, b) \in \mathbb{N} \times \mathbb{N} \}$, and we can define addition on $\mathbb{Z}$ by $[(a, b)]_\sim + [(c, d)]_\sim = [(a + c, b + d)]_\sim$, where $a + c$ and $b + d$ are computed using addition of natural numbers.

2. Show that addition is "well-defined" on $\mathbb{Z}$. That is, its definition does not really depend on which representatives you pick from each equivalence class: if $[(a,b)]_\sim = [(x,y)]_\sim$ and $[(c,d)]_\sim = [(u,v)]_\sim$, then $[(a,b)]_\sim + [(c,d)]_\sim = [(a+c,b+d)]_\sim = [(x+u,y+v)]_\sim = [(x,y)]_\sim + [(u,v)]_\sim$. *[3]*

Solution. Suppose $[(a,b)]_\sim = [(x,y)]_\sim$ and $[(c,d)]_\sim = [(u,v)]_\sim$. By the definition of equivalence classes for $\sim$, this means that $(a,b) \sim (x,y)$ amd $(c,d) \sim (u,v)$, which, by the definition of $\sim$, means that $a+y = x+b$ and $c+v = u+d$, and hence that $(a+y)+(c+v) = (x+b)+(u+d)$. With a little help from the commutativity and associativity of $+$ for $\mathbb{N}$, it now follows that

$$(a+c)+(y+v) = (a+y)+(c+v) = (x+b)+(u+d) = (x+u)+(b+d),$$

so $(a+c,b+d) \sim (x+u,y+v)$ by the definition of $\sim$, and thus $[(a+c,b+d)]_\sim = [(x+u,y+v)]_\sim$. It follows that

$$[(a,b)]_\sim + [(c,d)]_\sim = [(a+c,b+d)]_\sim = [(x+u,y+v)]_\sim = [(x,y)]_\sim + [(u,v)]_\sim\,,$$

as required. ■

3. Define subtraction and multiplication on $\mathbb{Z}$. *[3]*

Solution. Define subtraction on $\mathbb{Z}$ by $[(a,b)]_\sim - [(c,d)]_\sim = [(a+d,b+c)]_\sim$. Informally, using the idea that $[(x,y)]_\sim$ represents the difference $x-y$, this definition amounts to noticing that $(a-b)-(c-d) = (a+d)-(b+c)$.

Define multiplication on $\mathbb{Z}$ by $[(a,b)]_\sim \cdot [(c,d)]_\sim = [(ac+bd,ad+bc)]_\sim$. Informally, this boils down to $(a-b)(c-d) = (ac+bd)-(ad+bc)$.

One ought to check that both of these definitions are also "well-defined" before using them, but I didn't ask for that ... :-) ■