**Mathematics 2200H – Mathematical Reasoning**
TRENT UNIVERSITY, Fall 2017

**Solutions to Assignment #5**
**The natural numbers**

Recall from class that we defined the natural numbers from the empty set, $\emptyset$, and the *successor function*, $S(x) = x \cup \{x\}$, as follows:

- $0 = \emptyset$
- Given that the natural number $n$ has been defined, the next natural number (which we usually call $n+1$) is $S(n)$.

This definition makes each natural number be the set of all of its predecessors:

$0 = \emptyset$ has no predecessors
$1 = S(0) = 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{0\}$
$2 = S(1) = 1 \cup \{1\} = \{0, 1\}, \ldots$
$\vdots$
$n+1 = S(n) = n \cup \{n\} = \{0, 1, \ldots, n\}$
$\vdots$

One can proceed to define the usual arithmetic operations: addition by recursion from the successor function – $n+0 = n$ and, given that $n+k$ has been defined, $n+S(k) = S(n+k)$ – then multiplication by recursion from addition in a similar way, and so on.

**1.** Use induction to show that addition of the natural numbers, if defined from the successor function as above, is commutative (that is, $n+k = k+n$). *[5]*

SOLUTION. *[The big problem with this one is having to do one induction within another induction and keeping track of it all ... ]* We proceed by induction on $k$ to show that $n+k = k+n$ for all $n, k \in \mathbb{N}$:

*Base Step ($k=0$):* By the definition of $+$, $n+0 = n$ for all $n \in \mathbb{N}$. To show that $0+n = n$ as well, we prove a subsidiary result:

LEMMA 1. $0 + m = m$ for all $m \in \mathbb{N}$.

PROOF. We proceed by induction on $m$:

*Base Step ($m = 0$):* $0 + 0 = 0$ by the definition of $+$.

*Inductive Hypothesis:* Assume that $0 + m = m$.

*Inductive Step ($m \to S(m)$):* [To show: $0 + S(m) = S(m)$.] By the definition of $+$, $0 + S(m) = S(0+m)$; since the inductive hypothesis tells us that $0+m = m$, it follows that $0 + S(m) = S(0+m) = S(m)$, as is required to show.

It follows by induction that $0 + m = m$ for all $m \in \mathbb{N}$. $\square$

Applying the lemma above with $m = n$, it follows that $n + 0 = n = 0 + n$ for all $n \in \mathbb{N}$, as required to show at the base step ($k = 0$) of the induction on $k$.

*Inductive Hypothesis:* Assume that $n + k = k + n$ for all $n \in \mathbb{N}$.

*Inductive Step ($k \to S(k)$):* [To show: $n+S(k) = S(k)+n$ for all $n \in \mathbb{N}$.] By the definition of $+$, $n + S(k) = S(n + k)$; since the inductive hypothesis tells us that $n + k = k + n$, it follows that $n + S(k) = S(n + k) = S(k + n) = k + S(n)$. To finish the job, we need to get that $k + S(n) = S(k) + n$; we do so by proving another subsidiary result:

LEMMA 2. $a + S(b) = S(a) + b$ for all $a$, $b \in \mathbb{N}$.

PROOF. We proceed by induction on $b$:

*Base Step ($b = 0$):* For any $a \in \mathbb{N}$, using the definition of $+$ repeatedly gives us $a + S(0) = S(a + 0) = S(a) = S(a) + 0$.

*Inductive Hypothesis:* $a + S(b) = S(a) + b$ for all $a \in \mathbb{N}$.

*Inductive Step ($b \to S(b)$):* [To show: $a + S(S(b)) = S(a) + S(b)$.] Suppose $a \in \mathbb{N}$. By the definition of $+$, $a+S(S(b)) = S(a + S(b))$; applying the inductive hypothesis tells us that $S(a + S(b)) = S(S(a) + b)$; applying the definition of $+$ again gives $S(S(a) + b) = S(a) + S(b)$. Thus $a + S(b) = S(a) + b$, as is required to show.

It follows by induction that $a + S(b) = S(a) + b$ for all $a$, $b \in \mathbb{N}$. □

Applying the lemma above with $a = k$ and $b = n$ now tells us that $k + S(n) = S(k) + n$, and so $n + S(k) = S(n + k) = S(k + n) = k + S(n) = S(k) + n$, as we were required to show at the inductive step of our induction on $k$.

It follows by induction that $n + k = k + n$ for all $n$, $k \in \mathbb{N}$. ■

**2.** Use the definition of the natural numbers given above to define $<$ on the natural numbers. *[1]*

SOLUTION. Since each natural number is the set of all of its predecessors, both $n < k \iff n \in k$ and $n < k \iff n \subsetneq k$ work as possible definitions of $<$. ■

**3.** Verify that your definition of $<$ on the natural numbers is a (strict) linear order. *[4]*

NOTE. That is, you must check that $<$ is irreflexive ($n \not< n$) and transitive ($k < m$ and $m < n$ imply $k < n$), and satisfies trichotomy (exactly one of $n < m$, $n = m$, or $m < n$ must hold).

SOLUTION. Suppose we defined $<$ on $\mathbb{N}$ by $n < k \iff n \subsetneq k$. Then $<$ satisfies

- irreflexivity because $A \subsetneq A$ for any set $A$, since $A = A$ for any set;
- transitivity because $A \subsetneq B$ and $B \subsetneq C$ implies that $A \subsetneq C$ for any sets $A$, $B$, and $C$ [think about what $\subsetneq$ means ... ]; and
- trichotomy because if $n = m$, then neither $n \subsetneq m$ or $m \subsetneq n$ can be true, and if $n \neq m$, the one of $n \subsetneq m$ or $m \subsetneq n$ must be true: whichever of $n$ or $m$ occurs first in constructing the natural numbers is a subset of the other, since all of its predecessors are also predecessors of the other, *i.e.* all of its elements are also elements of the other. (It should be obvious that $n \subsetneq m$ and $m \subsetneq n$ can't both be true.)

Thus $<$ is a strict linear order. ■