Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2017

Solutions to Assignment #4
Induction

Recall that $n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$.

1. Use induction to show that $\dfrac{n^n}{3^n} < n! < \dfrac{n^n}{2^n}$ for all $n \geq 6$. *[5]*

Note. You could think of this result as a first, crude, cut at Stirling's Formula, which lets one approximate $n!$ for large n.

Solution. We have two inequalities to verify, which we will do separately.

First, the easier one, $n! < \dfrac{n^n}{2^n}$ for all $n \geq 6$, which we will do by induction on n:

Base Step $(n = 6)$: $6! = 720 < 729 = 3^6 = \left(\dfrac{6}{2}\right)^6 = \dfrac{6^6}{2^6}$, as required.

Inductive Hypothesis: $n! < \dfrac{n^n}{2^n}$

Inductive Step $(n \to n+1)$: Using the inductive hypothesis and much algebra:

$$\begin{aligned}
(n+1)! = n! \cdot (n+1) &< \frac{n^n}{2^n} \cdot (n+1) = 2 \cdot \frac{n^n}{2^n} \cdot \frac{n+1}{2} = \left[\frac{n^n}{2^n} + \frac{n^n}{2^n}\right] \cdot \frac{n+1}{2} \\
&= \left[\frac{n^n}{2^n} + \frac{n \cdot n^{n-1}}{2^n}\right] \cdot \frac{n+1}{2} = \left[\frac{n^n + \binom{n}{1} \cdot n^{n-1}}{2^n}\right] \cdot \frac{n+1}{2} \quad \left[\text{Since } \binom{n}{1} = n.\right] \\
&< \left[\frac{n^n + \binom{n}{1} \cdot n^{n-1} + \binom{n}{2} n^{n-2} + \cdots + \binom{n}{n-1} n + 1}{2^n}\right] \cdot \frac{n+1}{2} \\
&= \frac{(n+1)^n}{2^n} \cdot \frac{n+1}{2} \quad [\text{Using the binomial formula.}] \\
&= \frac{(n+1)^{n+1}}{2^{n+1}}, \text{ as required.}
\end{aligned}$$

It follows by induction that $n! < \dfrac{n^n}{2^n}$ for all $n \geq 6$. $\square$

Second, we tackle the harder inequality, $\dfrac{n^n}{3^n} < n!$ for all $n \geq 6$. This, like the previous inequality, can be obtained in many ways; the inductive method used here sticks to fairly low-power algebra for the most part. Before tackling the induction, it's useful to isolate two facts that we'll need rather than have their statement and explanation clutter up the inductive step.

Fact I: $\dbinom{n}{k} n^{n-k} < \dfrac{n^n}{k!}$ for $0 \leq k \leq n$.

Explanation: $\binom{n}{k} n^{n-k} = \frac{n!}{(n-k)!k!} n^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{k!} n^{n-k} < \frac{n^k}{k!} n^{n-k} = \frac{n^n}{k!}$. $\square$

Fact II: For any $n \geq 0$, $\sum_{k=0}^{n} \frac{1}{k!} < e = 2.7182\ldots$

Explanation: You may remember, most likely from calculus, that $e^x = 1 + x + \frac{x}{2} + \frac{x^2}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Plugging in $x = 1$ gives $e = \sum_{k=0}^{\infty} \frac{1}{k!}$, from which $e > \sum_{k=0}^{n} \frac{1}{k!}$ follows, since every term $\frac{1}{k!} > 0$. $\square$

We shall now show that $\dfrac{n^n}{3^n} < n!$ for all $n \geq 6$ by induction on n.

Base Step ($n = 6$): $\frac{6^6}{3^6} = \left(\frac{6}{3}\right)^6 = 2^6 = 64 < 720 = 6!$, as required.

Inductive Hypothesis: $\dfrac{n^n}{3^n} < n!$

Inductive Step ($n \to n+1$): We'll use the two facts, the binomial theorem, and some algebra:

$$\begin{aligned}
\frac{(n+1)^{n+1}}{3^{n+1}} &= \frac{n+1}{3^{n+1}}(n+1)^n = \frac{n+1}{3^{n+1}} \sum_{k=0}^{n} \binom{n}{k} n^{n-k} \quad \text{[Using the binomial theorem.]} \\
&< \frac{n+1}{3^{n+1}} \sum_{k=0}^{n} \frac{n^n}{k!} \quad \text{[Using Fact I.]} \\
&= \frac{(n+1)n^n}{3^n \cdot 3} \sum_{k=0}^{n} \frac{1}{k!} \\
&< \frac{n^n}{3^n} \cdot \frac{n+1}{3} \cdot e \quad \text{[Using Fact II.]} \\
&< n! \cdot (n+1) \cdot \frac{e}{3} \quad \text{[Using the inductive hypothesis.]} \\
&< n! \cdot (n+1) \quad \left[\text{Since } \frac{e}{3} < 1.\right] \\
&= (n+1)!, \text{ as required.}
\end{aligned}$$

It follows by induction that $\dfrac{n^n}{3^n} < n!$ for all $n \geq 6$. $\blacksquare$

2. Without looking it up, try to find or guess a formula for the sum of the first n cubes,

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^2 + 3^3 + \cdots + (n-1)^3 + n^3 \,,$$

and then use induction to verify that your formula is true. *[5]*

Hint: This has something a little surprising to do with the fact that

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \,.$$

Solution. As it turns out, $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.* This is pretty straightforward to verify by induction on n:

Base Step $(n = 1)$: $\left[\frac{1(1+1)}{2}\right]^2 = \left[\frac{2}{2}\right]^2 = 1^2 = 1 = 1^3 = \sum_{i=1}^{1} i^3$, as required.

Inductive Hypothesis: $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Inductive Step $(n \to n+1)$: First, using the inductive hypothesis and some algebra:

$$\sum_{i=1}^{n+1} i^3 = \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \frac{n^2\left(n^2+2n+1\right)}{4} + \frac{4(n+1)^3}{4}$$
$$= \frac{\left(n^4+2n^3+n^2\right) + 4\left(n^3+3n^2+3n+1\right)}{4} = \frac{n^4+6n^3+13n^2+12n+4}{4}$$

Second, just using some algebra:

$$\left[\frac{(n+1)\left((n+1)+1\right)}{2}\right]^2 = \left[\frac{(n+1)(n+2)}{2}\right]^2 = \frac{(n+1)^2(n+2)^2}{4}$$
$$= \frac{\left(n^2+2n+1\right)\left(n^2+4n+4\right)}{4} = \frac{n^4+6n^3+13n^2+12n+4}{4}$$

Combining these gives us $\displaystyle\sum_{i=1}^{n+1} i^3 = \frac{n^4+6n^3+13n^2+12n+4}{4} = \left[\frac{(n+1)\left((n+1)+1\right)}{2}\right]^2$, as required.

It follows by mathematical induction that $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$ for all $n \geq 1$. ■

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* This means that $(1+2+\cdots+n)^2 = 1^3 + 2^2 + \cdots + n^3$, which still blows my mind ... :-)