Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2017

Solutions to Assignment #2
Digits and Divisibility

Suppose $b \in \mathbb{N}$ and $b > 1$. If n is any positive natural number, then there are unique natural numbers k and d_k, d_{k-1}, ... , d_0, such that $0 \leq d_i < b$ for each i with $0 \leq i \leq k$, $d_k \neq 0$, and $n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_0 b^0$. The base b representation of n is then the sequences of digits $d_k d_{k-1} \dots d_0$; if we need to emphasize that it is in base b, we will often write something like $(d_k d_{k-1} \dots d_0)_b$. For example, consider $533 = (533)_{10}$. 533 would be written as 1000010101 in base 2 because $533 = 512 + 16 + 4 + 1 = 1 \cdot 2^9 + 0 \cdot 2^8 + 0 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$.

1. Suppose $n = (d_k d_{k-1} \dots d_1 d_0)_{10}$. Show that $3 \mid n \iff 3 \mid (d_k + d_{k-1} + \cdots + d_1 + d_0)$. For which natural numbers > 1 other than 3 does this trick work? *[5]*

SOLUTION. Here goes:

$$\begin{aligned}
3 \mid n &\iff n \equiv 0 \pmod{3} \\
&\iff (d_k d_{k-1} \dots d_0)_{10} \equiv 0 \pmod{3} \\
&\iff \left(d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_1 10^1 + d_0 10^0\right) \equiv 0 \pmod{3} \\
&\iff \left(d_k 1^k + d_{k-1} 1^{k-1} + \cdots + d_1 1^1 + d_0 1^0\right) \equiv 0 \pmod{3} \text{ [Since } 10 \equiv 1 \pmod{3}.\text{]} \\
&\iff (d_k + d_{k-1} + \cdots + d_1 + d_0) \equiv 0 \pmod{3} \\
&\iff 3 \mid (d_k + d_{k-1} + \cdots + d_1 + d_0) \qquad \square
\end{aligned}$$

2. Given a base b, for which integers c is it true that whenever $n = (d_k d_{k-1} \dots d_1 d_0)_b$, we have $c \mid n \iff c \mid (d_k + d_{k-1} + \cdots + d_1 + d_0)$? *[2]*

SOLUTION. To use the argument given in the solution to **1** above, with 3 replaced by c and base 10 replaced by base b, all you need is to have $b \equiv 1 \pmod{c}$. $\square$

3. Devise a similar trick for testing for divisiblity by 11 in base 10. *[3]*

SOLUTION. Note that $10 \equiv -1 \pmod{11}$. This means that:

$$\begin{aligned}
11 \mid n &\iff n \equiv 0 \pmod{11} \\
&\iff (d_k d_{k-1} \dots d_0)_{10} \equiv 0 \pmod{11} \\
&\iff \left(d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_1 10^1 + d_0 10^0\right) \equiv 0 \pmod{11} \\
&\iff \Big(d_k (-1)^k + d_{k-1} 9_1)^{k-1} + \cdots + d_1 (-1)^1 + d_0 (-1)^0\Big) \equiv 0 \pmod{11} \\
&\iff \left(d_0 - d_1 + d_2 - \cdots + (-1)^k d_k\right) \equiv 0 \pmod{11} \\
&\iff 11 \mid \left(d_0 - d_1 + d_2 - \cdots + (-1)^k d_k\right)
\end{aligned}$$

This means that 11 divides n, written out in base 10, if and only if 11 divides the number obtained by alternately adding and subtracting all the digits of n. $\square$