Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2017

Solutions to Assignment #10
More counting

A set A is said to be *countable* if $|A| \leq |\mathbb{N}|$, and *countably infinite* if $|A| = |\mathbb{N}|$.

1. Suppose A_n, $n \in \mathbb{N}$, is a countably infinite collection of disjoint countably infinite sets. (So each A_n is countably infinite and $A_m \cap A_k = \emptyset$ whenever $k \neq m$.) Show that $A = \bigcup_{n=0}^{\infty} A_n$ is also countably infinite. *[4]*

SOLUTION. Since each A_n is countably infinite, it can be enumerated, *i.e.* $A_n = \{ a_0^n, a_1^n, a_2^n, \dots \}$.▌ Since the A_ns are disjoint, we can assume that $a_k^n = a_\ell^n$ only when $n = m$ and $k = \ell$. Note that every a_k^n is an element of $A = \bigcup_{n=0}^{\infty} A_n$, and every element of A must be a_k^n for some unique n, $k \in \mathbb{N}$

As is noted in the text (see p. 351 in §8.2), the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f(x,y) = x + \dfrac{(x+y)^2 + (x+y)}{2}$ is both 1–1 and onto. We will use it to define a function g from $A = \bigcup_{n=0}^{\infty} A_n$ to $\mathbb{N}$ by $g\left(a_k^n\right) = f(n,k)$. By the observation in the paragraph above, g is defined for all elements of A. We claim that g is both 1–1 and onto.

g is 1–1: If $a_k^n \neq a_\ell^m$, then $(n.k) \neq (m, \ell)$, so $g\left(a_k^n\right) = f(n,k) \neq f(m,\ell) = g\left(a_\ell^m\right)$ because f is 1–1. Thus g is 1–1.

g is onto: Suppose $m \in \mathbb{N}$. Since $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is onto, there are n and k in $\mathbb{N}$ such that $f(n,k) = m$, but then $g\left(a_k^n\right) = f(n,k) = m$. Thus g is onto.

Since there is a 1–1 onto function $g : A \to \mathbb{N}$, $|A| = |\mathbb{N}|$, A is countably infinite. ■

2. Suppose C is an infinite subset of a countably infinite set D. Show that C is also countably infinite.

SOLUTION. Since D is countably infinite, there is a 1–1 onto function $f : \mathbb{N} \to D$. We will define a sequence n_k of natural numbers as follows:

- Let n_0 be the least $n \in \mathbb{N}$ such that $f(n) \in C$.
- Given that $n_k \in \mathbb{N}$ has been defined for some $k \in \mathbb{N}$, let n_{k+1} be the least $n > n_k$ such that $f(n) \in C$.

Note that because C is infinite, there is always another n_{k+1} to be found.

Now define $g : \mathbb{N} \to C$ by $g(k) = f\left(n_k\right)$. g is 1–1 because it is the composition of two 1–1 functions: $k \mapsto n_k$ is 1–1 since $n_k < n_{k+1}$ for every k, and f was already assumed to be 1-1. g is onto because f enumerates all of D, $C \subseteq D$, and the sequence of n_ks is defined precisely to capture all the natural numbers that f uses to index elements of C. Thus $|\mathbb{N}| = |C|$, *i.e.* C is countably infinite. ■

3. Suppose A is countable and there is an onto function $F : A \to B$. Show that B is countable. *[3]*

SOLUTION. Since A is countable, *i.e.* $|A| \leq |\mathbb{N}|$, there is a 1–1 function $f : A \to \mathbb{N}$. Define a function $g : B \to \mathbb{N}$ by $g(b) = n$ for the least $n \in \mathbb{N}$ such that $F(a) = b$ and $f(a) = n$ for some $a \in A$; that is, $g(b) = \min\{\, f(a) \mid a \in A \text{ and } F(a) = b \,\}$. (Note that F being onto guarantees there is at least one a such that $F(a) = b$.) We claim that g is 1–1:

Suppose b, $c \in B$ and $b \neq c$. Let a, $a' \in A$ be the elements such that $F(a) = b$ and $F(a') = c$, with $g(b) = f(a)$ and $g(c) = f(a')$ per the definition above. Since $F(a) = b \neq c = F(a')$, we must have $a \neq a'$, but then $f(a) \neq f(a')$ because f is 1–1, so $g(b) = f(a) \neq f(a') = g(b)$. Thus g is 1–1.

Since there is a 1–1 function $g : B \to \mathbb{N}$, we have that $|B| \leq |\mathbb{N}|$ by definition, *i.e.* B is countable. ■