Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2017

Solutions to Assignment #1
Sir! Yes, sir! General Formula, sir!

Consider the following:

PROBLEM. *Find a general formula for two squares whose sum* $= 2$.

What does this problem ask for? Presumably, a formula (or formulas) for u and v that generates all the possible u and v such that $u^2 + v^2 = 2$. One complication is that the possible answers differ depending on just what kind of numbers u and v are allowed to be and what operations and functions we are allowed to use in the formulas. In what follows, unless stated otherwise, please stick to the basic arithmetic and algebraic operations (that is, $+$, $-$, $\cdot$, $/$, powers, and roots) and the specified number systems.

1. Solve the problem assuming that u and v are required to be integers. *[1]*

SOLUTION. The only integers whose squares are less than or equal to 2 are -1, 0, and 1. Since $1^1 = (-1)^2 = 1$ and $0^2 = 0$, it is not hard to see that if $u^2 + v^2 = 2$ when u and v are integers, then $u = \pm 1$ and $v = \pm 1$. This means that there are only four solutions for (u, v) such that $u^2 + v^2 = 1$, namely$(1, 1)$, $(-1, 1)$, $(1, -1)$, and $(-1, -1)$, respectively. □

2. Solve the problem assuming that u and v are only required to be real numbers. *[2]*

SOLUTION. It's pretty obvious that we could plug in any real number u with $u^2 \leq 2$ and then solve for v: $v^2 = 1 - u^2$, so $v = \pm\sqrt{1 - u^2}$. This would mean that one general solution would be $-\sqrt{2} \leq u \leq \sqrt{2}$ and $v = \pm\sqrt{1 - v^2}$.

Interchanging the roles of u and v would give a similar general solution: $-\sqrt{2} \leq v \leq \sqrt{2}$ and $u = \pm\sqrt{1 - v^2}$. Other solutions are possible, too; any solution to **3** is also a solution to **2**, for example. □

3. Solve the problem without using roots or fractional powers assuming that u and v are only required to be real numbers. *[5]*

SOLUTION. The original problem is problem 1 from Lewis Carroll's *Pillow Problems*. Here is his solution:

> Let u, v be the Nos.
> Then $u^2 + v^2 = 2$.
> Evidently '$(1 + k)$, $(1 - k)$' is a form for the squares.
> Also, if we write '$2m^2$' for '2' (which will not interfere with the problem, as we can divide m^2, and get $\frac{u^2}{m^2} + \frac{v^2}{m^2} = 2$, the above form becomes '$\left(m^2 + k\right)$, $\left(m^2 - k\right)$'.
> Now, as these are *squares*, their resemblance to
> $$\text{‘}\left(a^2 + b^2 + 2ab\right),\ \left(a^2 + b^2 - 2ab\right)\text{’}$$

at once suggests itself; so that the problem depends on the known one of finding a, b, such that $\left(a^2+b^2\right)$ is a square; and we can then take $2ab$ as k.

A general form of this is

$$\begin{aligned} a &= x^2 - y^2\,, \\ b &= 2xy\,; \\ \therefore a^2 + b^2 &= \left(x^2+y^2\right)^2\;; \end{aligned}$$

$\therefore$ the formula '$u^2 + v^2 = 2m^2$' becomes

$$\left(x^2-y^2+2xy\right)^2 + \left(x^2-y^2-2xy\right)^2 = 2\left(x^2+y^2\right)^2\;;$$

$$i.e.\quad \left(\frac{x^2-y^2+2xy}{x^2+y^2}\right)^2 + \left(\frac{x^2-y^2-2xy}{x^2+y^2}\right)^2 = 2\,.$$

Q.E.F.

That is, a general solution to $u^2+v^2=2$, for u and v to be real numbers is to have $u = \frac{x^2-y^2+2xy}{x^2+y^2}$ and $v = \frac{x^2-y^2-2xy}{x^2+y^2}$, where x and y can be any real numbers (except that you can't have $x=y=0$, lest you divide by 0, a point that Carroll ignores). Note that Carroll describes the thought process leading to the solution, which does rely on some background knowledge (or additional work), particularly that $\left(x^2-y^2\right)^2+(2xy)^2 = \left(x^2+y^2\right)^2$. While he does not verify that the solution works, which is easy with a little algebra, he doesn't really need to because backtracking through his solution will also verify that it does. □

4. How do your answers to **2** and **3** change if u and v could be complex numbers? *[2]*

Solution. In the solution to **2**, allowing complex numbers removes the restriction that $-\sqrt{2} \le u \le \sqrt{2}$, since $v = \pm\sqrt{2-u^2}$ still makes sense even if $2-u^2$ is a negative real number or is a complex number. In the solution to **3**, nothing really changes: the algebra still works out even if x and y, and hence u and v, are complex numbers. Note that you do have to keep the restriction that you can't have $x=y=0$. ■