Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2017
Take-Home Final Examination
Due on ~~Friday~~ Saturday, 16 December, 2017.

Instructions: Do both of parts **P** and **Q**, and, if you wish, part **R** as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part Prime. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Give the truth tables of all sixteen binary logical connectives, and find a logically equivalent formula for each one using only the connectives $\vee$ and $\neg$. (If you don't already know a symbol for one of the sixteen, just invent one . . . :-)

2. Suppose the sequence c_n is defined recursively as follows:
- Let $c_0 = 0$.
- Assuming c_n has been defined, let $c_{n+1} = 1 + \dfrac{c_n}{2}$.

Show that $c_n = \dfrac{2^n - 1}{2^{n-1}}$ for all $n \geq 0$.

3. Suppose $\mathcal{C} = \{\, A \subseteq \mathbb{N} \mid \mathbb{N} \setminus A = \{n \in \mathbb{N} \mid n \notin A\} \text{ is finite} \,\}$ is the collection of *cofinite* subsets of $\mathbb{N}$. Show that $\mathcal{C}$ is countably infinite, *i.e.* $|\mathcal{C}| = |\mathbb{N}|$.

4. Suppose the real numbers are defined, as was done in class, as equivalence classes of Cauchy sequences of real numbers. For your convenience:

> A sequence $\{q_n\}$ is *Cauchy* if for every $\epsilon > 0$, there is an N such that for all m, $k \geq N$, $|q_m - q_k| < \varepsilon$. Define an equivalence relation $\approx$ on Cauchy sequences of rationals by $\{q_n\} \approx \{p_n\}$ if for all $\varepsilon > 0$, there is an N such that for all $n \geq N$, $|q_n - p_n| < \varepsilon$. Then $\mathbb{R} = \big\{\, [\{q_n\}]_\approx \mid \{q_n\} \text{ is a Cauchy sequence of rationals} \,\big\}$.

Define multiplication on the real numbers using this definition and check that multiplication is commutative.

Part Qrime: Do any *four* (4) of problems **5** – **11**. *[40 = 4 × 10 each]*

5. Suppose a, $r \in \mathbb{R}$. Define the sequence $\{a_n\}$ recursively as follows:
- Let $a_0 = a$.
- Assuming a_n has been defined, let $a_{n+1} = ra_n + a$.

Find a closed formula for a_n in terms of a, r, and n.

6. Let $\mathbb{A} = \{\, a \in \mathbb{R} \mid a \text{ is a root of a polynomial with rational coefficients} \,\}$. Show that $\mathbb{A}$ is countable.

More questions on page 2 . . .

... and here they are:

7. Recall that a strict partial order is a binary relation that is irreflexive and transitive, but does not necessarily satisfy trichotomy. Suppose $\triangleleft$ is a strict partial order on a finite set S. Show that there is a strict linear order, call it $\blacktriangleleft$, on S such that $s \triangleleft t$ implies $s \blacktriangleleft t$ for all s, $t \in S$.

8. What is the maximum area of a square that can be fitted inside a unit cube? [The square need not be parallel to any face of the cube.]

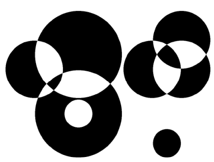
9. The inhabitants of the Island of Knights and Knaves are either knights, who always tell the truth, or knaves, who always lie. While visiting the Island you encounter nine inhabitants: Bozo, Zoey, Marge, Bart, Zed, Joe, Dave, Sue and Mel. Bozo claims that it's false that Dave is a knave. Zoey tells you that Marge is a knave and Sue is a knight. Marge claims that Bozo is a knave. Bart claims, "Zed could claim that I am a knight." Zed says that only a knave would say that Dave is a knave. Joe says, "It's not the case that Bart is a knave." Dave claims that Mel and Joe are both knights or both knaves. Sue tells you that Zed is a knave. Mel claims, "Neither Bozo nor Zoey are knaves." Determine, as completely as you can, whether each of the nine inhabitants is a knight or a knave.

10. Show that every natural number n is equal to a sum of the form

$$a_k \cdot k! + a_{k-1} \cdot (k-1)! + \cdots + a_2 \cdot 2! + a_1 \cdot 1! + a_0 \cdot 0!$$

for some $k \geq 0$ and such that each a_i is a natural number with $0 \leq a_i \leq i$.

11. Suppose that finitely many, possibly overlapping, circles are drawn in the plane, dividing it into regions whose borders are made up of circular arcs. Show that the regions can be coloured using white and black so that no two regions sharing a common border have the same colour. [Just in case: touching at finitely many points does not constitute sharing a border.]



For maximum credit, give a solution that does not use induction.

[Total = 80]

Part Rrime! More bonus ...

α. Write an original poem about logic or mathematics. *[1]*

β. Draw a picture to explain why problem **11** could be a Mickey Mouse problem.* *[1]*

ENJOY THE BREAK EVEN MORE THAN YOU ENJOYED THIS COURSE!

* With apologies to Walt Disney. ...