Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2017

Assignment #8
The unkindest cut of all?
Due on Thursday, 9 November.

We defined the set of real numbers in class using equivalence classes of Cauchy sequences of rational numbers. This definition makes it fairly easy to define the basic arithmetic operations, at the cost of being tedious – though pretty easy – to check that definitions work and have the usual algebraic properties. It's a bit harder to define $<$ on the real numbers and show it has the usual prperties using this approach, though. The main alternate method for defining the real numbers, using *schnitts* or *Dedekind cuts*, makes it fairly easy to define $<$ and establish its properties, but at the cost of making the definition of the arithmetic operations (and obtaining their basic properties) somewhat more cumbersome.

DEFINITION. A *schnitt* or *Dedekind cut* is a subset $S \subseteq$ of the rational numbers satisfying the following conditions:

i. $S \neq \emptyset$ and $S \neq Q$.
ii. S has no greatest element, *i.e.* if $p \in S$, then there is a $q \in S$ such that $p < q$.
iii. S is closed downward, *i.e.* if $p \in S$ and $r \in \mathbb{Q}$ with $r < p$, then $r \in S$.

Using schnitts, the set of real numbers is simply the collection of all schnitts, *i.e.* $\mathbb{R} = \{\, S \mid S \text{ is a schnitt} \,\}$. The linear order $<$ on the real numbers is then defined by $S < T$ if and only if $S \subsetneq T$.

Intuitively, each real number r corresponds to the schnitt $R = \{\, p \in \mathbb{Q} \mid p < r \,\}$.

1. Show that $<$, as defined above, is a strict linear order on $\mathbb{R}$. *[5]*

Recall from somewhen before calculus that a set X of real numbers has an upper (respectively, lower) bound if there is a real number which is greater (respectively, less) than or equal to every real number in X. A least upper bound for X is an upper bound for X that is less than or equal to every upper bound for X. (Similarly, a greatest lower bound for X is ...)

2. Suppose that a set $X \neq \emptyset$ of real numbers (using the schnitt definition above) has an upper bound. Show that X has a least upper bound. *[5]*