Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2017

Assignment #6
The integers

Due on Thursday, 19 October.

Recall from class that we defined the natural numbers from the empty set, $\emptyset$, and the *successor function*, $S(x) = x \cup \{x\}$, and then proceeded to define addition of natural numbers by recursion from the successor function: $n + 0 = n$ and, given that $n + k$ has been defined, $n + S(k) = S(n + k)$. Multiplication of natural numbers was then defined by recursion from addition in a similar way. We also defined the *predecessor function*, $P(0) = 0$ and $P(S(k)) = k$, and used it to define a difference function, $D(a, b) = a - b$ if $a > b$ and $D(a, b) = 0$ otherwise, which is as close as you can get to proper subtraction without having negative numbers. This assignment is concerned with building the set of integers, $\mathbb{Z}$, from the natural numbers using equivalence relations.

DEFINITION. Let $\mathbb{N} \times \mathbb{N} = \{ (a, b) \mid a, b \in \mathbb{N} \}$ be the collection of all ordered pairs of natural numbers. Define a binary relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ by letting $(a, b) \sim (c, d)$ if and only if $a + d = c + b$.

Informally, $(a, b) \sim (c, d)$ exactly when $a - b = c - d$. (This has to be informal at the moment since we don't have real subtraction yet because the natuiral numbers don't include negatives.)

1. Show that $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$. *[4]*

DEFINITION. Denote the equivalence class of $(a, b) \in \mathbb{N} \times \mathbb{N}$ by $[(a, b)]_\sim$. Then $\mathbb{Z} = \{ [(a, b)]_\sim \mid (a, b) \in \mathbb{N} \times \mathbb{N} \}$, and we can define addition on $\mathbb{Z}$ by $[(a, b)]_\sim + [(c, d)]_\sim = [(a + c, b + d)]_\sim$, where $a + c$ and $b + d$ are computed using addition of natural numbers.

2. Show that addition is "well-defined" on $\mathbb{Z}$. That is, its definition does not really depend on which representatives you pick from each equivalence class: if $[(a, b)]_\sim = [(x, y)]_\sim$ and $[(c, d)]_\sim = [(u, v)]_\sim$, then $[(a, b)]_\sim + [(c, d)]_\sim = [(a + c, b + d)]_\sim = [(x + u, y + v)]_\sim = [(x, y)]_\sim + [(u, v)]_\sim$. *[3]*

3. Define subtraction and multiplication on $\mathbb{Z}$. *[3]*