Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2017

Assignment #5
The natural numbers
Due on Thursday, 12 October.

Recall from class that we defined the natural numbers from the empty set, $\emptyset$, and the *successor function*, $S(x) = x \cup \{x\}$, as follows:

- $0 = \emptyset$
- Given that the natural number n has been defined, the next natural number (which we usually call $n+1$) is $S(n)$.

This definition makes each natural number be the set of all of its predecessors:

$$\begin{aligned}
&0 = \emptyset \text{ has no predecessors}\\
&1 = S(0) = 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{0\}\\
&2 = S(1) = 1 \cup \{\,1\,\} = \{\,0,\,1\,\}, \ldots\\
&\qquad\qquad\vdots\\
&n+1 = S(n) = n \cup \{n\} = \{\,0,\,1,\,\ldots,\,n\,\}\\
&\qquad\qquad\vdots
\end{aligned}$$

One can proceed to define the usual arithmetic operations: addition by recursion from the successor function – $n+0=n$ and, given that $n+k$ has been defined, $n+S(k)=S(n+k)$ – then multiplication by recursion from addition in a similar way, and so on.

1. Use induction to show that addition of the natural numbers, if defined from the successor function as above, is commutative (that is, $n+k=k+n$). *[5]*

2. Use the definition of the natural numbers given above to define $<$ on the natural numbers. *[1]*

3. Verify that your definition of $<$ on the natural numbers is a (strict) linear order. *[4]*

NOTE. That is, you must check that $<$ is irreflexive ($n \not< n$) and transitive ($k<m$ and $m<n$ imply $k<n$), and satisfies trichotomy (exactly one of $n<m$, $n=m$, or $m<n$ must hold).