Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2016

Solutions to Assignment #6
The integers defined via an equivalence relation

Here is the way, mentioned in class, of defining the integers from the natural numbers that is analogous to the way the rationals were defined in class from the integers.

Let $\mathbb{N}\times\mathbb{N} = \{\,(a,b) \mid a,\, b \in \mathbb{N}\,\}$ be the collection of all ordered pairs of natural numbers. Define the binary relation $\sim$ on $\mathbb{N}\times\mathbb{N}$ by setting $(a,b) \sim (c,d)$ if and only if $a+d = b+c$, where $+$ is the usual operation of addition on the natural numbers. Intuitively, $(a,b) \sim (c,d)$ exactly when $a-b = c-d$.

1. Verify that $\sim$ is an *equivalence relation* on $\mathbb{N}\times\mathbb{N}$. *[4]*

That is, you need to check the following:

i. $\sim$ is *reflexive*: $(a,b) \sim (a,b)$ for all $(a,b) \in \mathbb{N}\times\mathbb{N}$.

ii. $\sim$ is *commutative*: $(a,b) \sim (c,d)$ if and only if $(c,d) \sim (a,b)$ for all (a,b), $(c,d) \in \mathbb{N}\times\mathbb{N}$.

iii. $\sim$ is *transitive*: $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$ imply that $(a,b) \sim (e,f)$ for all (a,b), (c,d), $(e,f) \in \mathbb{N}\times\mathbb{N}$.

SOLUTION. Here goes!

i. [$\sim$ is reflexive.] If $(a,b) \in \mathbb{N}\times\mathbb{N}$, then $a+b = b+a$ by the commutativity of $+$ on $\mathbb{N}$, so $(a,b) \sim (a,b)$ by the definition of $\sim$.

ii. [$\sim$ is commutative.] Suppose (a,b), $(c,d) \in \mathbb{N}\times\mathbb{N}$. Then

$$\begin{aligned}
(a,b) \sim (c,d) \quad &\iff \quad a+d = b+c && \text{(By the definition of } \sim\text{.)}\\
&\iff \quad d+a = c+b && \text{(By the comutativity of = on } \mathbb{N}\text{.)}\\
&\iff \quad c+b = d+a && \text{(By the commutativity of =. :-)}\\
&\iff \quad (c,d) \sim (a,b) && \text{(By the definition of } \sim\text{.)}
\end{aligned}$$

so $(a,b) \sim (c,d)$ if and only if $(c,d) \sim (a,b)$, as desired.

iii. [$\sim$ is transitive.] Suppose (a,b), (c,d), $(e,f) \in \mathbb{N}\times\mathbb{N}$, $(a,b) \sim (c,d)$, and $(c,d) \sim (e,f)$. Then $a+d = b+c$ and $c+f = d+e$ by the definition of $\sim$. Adding these equations gives us $(a+d)+(c+f) = (b+c)+(d+e)$, in which we can rearrange both sides, using the commutativity and associativity of $+$ on $\mathbb{N}$, to get $(a+f)+(c+d) = (b+e)+(c+d)$. It follows, using the cancellation law for $+$ on $\mathbb{N}$, that $a+f = b+e$, so $(a,b) \sim (e,f)$ by the definition of $\sim$.

Since the binary relation $\sim$ is reflexive, commutative (also called *symmetric*), and transitive, it is an equivalence relation. ■

Given that $\sim$ is indeed an equivalence relation on $\mathbb{N}\times\mathbb{N}$, then the *equivalence class* of (a,b) is the set $[(a,b)]_\sim = \{\,(c,d) \in \mathbb{N}\times\mathbb{N} \mid (a,b) \sim (c,d)\,\}$, and we now define the set of integers to be $\mathbb{Z} = \{\,[(a,b)]_\sim \mid (a,b) \in \mathbb{N}\times\mathbb{N}\,\}$.

2. Define $+_{\mathbb{Z}}$, the operation of addition on the integers (defined as above), and show that it is associative. *[3]*

Solution. We define $+$ on $\mathbb{Z}$ as follows. For $[(a,b)]\,, [(c,d)] \in \mathbb{Z}$, we let

$$[(a,b)] +_{\mathbb{Z}} [(c,d)] = [(a+c, b+d)]\ .$$

To make sure this definition actually makes sense, we need to check that equivalence class we get as the sum does not depend on the choice of which ordered pairs we use from each equivalence class being summed. That is, if $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$, then we had better have $[(a+c,b+d)] = [(a'+c',b'+d')]$. This is mostly a matter of wading through the definitions:

$$\begin{aligned}
& [(a,b)] = [(a',b')] \text{ and } [(c,d)] = [(c',d')] \\
\Longrightarrow\ & (a,b) \sim (a',b') \text{ and } (c,d) \sim (c',d') \quad \text{(By the definition of equivalence classes.)} \\
\Longrightarrow\ & a+b' = b+a' \text{ and } c+d' = d+c' \quad \text{(By the definition of } \sim\text{.)} \\
\Longrightarrow\ & (a+b')+(c+d') = (b+a')+(d+c') \\
\Longrightarrow\ & (a+c)+(b'+d') = (b+d)+(a'+c') \quad \text{(Associativity and commutativity of } +_{\mathbb{N}}\text{.)} \\
\Longrightarrow\ & (a+c,b+d) \sim (a'+b',c'+d') \quad \text{(By the definition of } \sim\text{.)} \\
\Longrightarrow\ & [(a+c,b+d)] = [(a'+c',b'+d')] \quad \text{(By the definition of equivalence classes.)}
\end{aligned}$$

Thus $+_{\mathbb{Z}}$ is, as the professionals put it, "well-defined".

It remains to show that $+_{\mathbb{Z}}$ is associative. This is also largely a matter of wading through the definitions:

$$\begin{aligned}
([(a,b)] +_{\mathbb{Z}} [(c,d)]) +_{\mathbb{Z}} [(e,f)] &= [(a+c,b+d)] +_{\mathbb{Z}} [(e,f)] \quad \text{(By the definition of } +_{\mathbb{Z}}\text{.)} \\
&= [((a+c)+e,(b+d)+f)] \quad \text{(By the definition of } +_{\mathbb{Z}}\text{.)} \\
&= [(a+(c+e),b+(d+f))] \quad \text{(Associativity of } + \text{ on } \mathbb{N}\text{.)} \\
&= [(a,b)] +_{\mathbb{Z}} [(c+e,d+f)] \quad \text{(By the definition of } +_{\mathbb{Z}}\text{.)} \\
&= [(a,b)] +_{\mathbb{Z}} ([(c,d)] +_{\mathbb{Z}} [(e,f)]) \quad \text{(By the definition of } +_{\mathbb{Z}}\text{.)}
\end{aligned}$$

Thus $+_{\mathbb{Z}}$ is also associative. ■

3. Define $\cdot_{\mathbb{Z}}$, the operation of multiplication on the integers (defined as above), and show that it is commutative. *[3]*

Solution. For $[(a,b)]\,, [(c,d)] \in \mathbb{Z}$, we define $[(a,b)] \cdot_{\mathbb{Z}} [(c,d)] = [(ac+bd, ad+bc)]$. After this, of course, we need to check that $\cdot_{\mathbb{Z}}$ is well-defined and that it is commutative. Too long and boring – it's a lot like the solution to **2** – for your instructor ... ZZZZZZZZZ ■

In all of the above problems, you may assume that $+$ and $\cdot$ have been defined on $\mathbb{N}$, and have the usual algebraic properties.