Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2016

Solutions to Assignment #4
A little formality

Here is a formal definition of a fairly minimal first-order language for set theory:

The symbols of the language are as follows:

Variables: x_0, x_1, x_2, ...
Connectives: $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$
Quantifiers: $\forall$, $\exists$
Parentheses: (,)
Equality: $=$
Set Membership: $\in$ (a 2-place relation)
All of the above symbols are distinct, none is a substring of any other, and there are no other symbols in the language.

The formulas (*i.e.* statements) of the language are defined as follows:

1. For any variables x_i and x_j of the language, $(x_i = x_j)$ and $(x_i \in x_j)$ are formulas of the language.
2. If φ and ψ are any formulas of the language, then $(\neg\varphi)$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$ are also formulas of the language.
3. If φ is any formula of the language and x_i is any variable of the language, then $(\forall x_i \varphi)$ and $(\exists x_i \varphi)$ are also formulas of the language.
4. No string of symbols of the language is a formula of the language unless it was formed using (possibly many applications of) rules 1–3 above.

This language is inefficient in some ways – it could really use a symbol for the empty set and some additional relations, such as the subset relation, and overuses parentheses, among other things – but as first-order languages go it is pretty uncomplicated.

1. What are the possible lengths of formulas of the given language? *[5]*

Note: The length of a formula of the language is the number of symbols of the language making up the formula, counting repetitions. For example, each instance of a variable x_i counts as one symbol.

Solution. The shortest possible formulas in the given language are those of the form $(x_i = x_j)$ and $(x_i \in x_j)$, which each have five symbols. (All other formulas must be longer because they are built out of shorter formulas.) In particular, no formula of the language can have length 0, 1, 2, 3, or 4.

Given a formula φ of length n, it is possible to make formulas of length $n+3$, namely $(\neg\varphi)$, and of length $n+4$, such as $(\forall x_i \varphi)$. Combining this observation with the previous paragraph, it is easy to see that one can make formulas of all lengths of the form $n = 5 + 3k + 4\ell$, where k, $\ell \geq 0$. This means that there are formulas of length 5, 8, 9, 11, 12, 13, 14, ... (Note that we do get every length ≥ 11.)

It remains to show that there are no formulas of length 6, 7, or 10. No formula of length 6 or 7 could be formed by using connectives or quantifiers, because this would add

3 or more symbols to an existing formula, and there are no formulas of length 4 or less. By the same token, there can be no formula of length 10: it could not be formed by negating a formula because there is no formula of length 7, nor by adding a quantifier to an existing formula, because there is no formula of length 6, nor could it be formed by applying a binary connective to two shorter formulas because these would have to have a length of at least 5 each, so any formula with a binary connective would have at least 13 symbols.

Thus formulas of this length cannot have lengths of 0, 1, 2, 3, 4, 6, 7, or 10, but all other lemgths are possible. ■

2. Find a way to define ordered pairs in the given language. *[5]*

NOTE: The ordered pair (a, b) is different from the ordered pair (b, a) unless $a = b$. Your first problem for **2** is to figure out what it actually means to define such a concept in the given language.

SOLUTION. Someone figured this out in seminar, much to your instructor's surprise. Defining (a, b) to be $\{a, \{a, b\}\}$ works – I'll leave the details of why to you. ■