Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2016

Solutions to Assignment #3
A little pidgeon quibbled with my Aristotelian logic ...

Suppose we have a first-order language with the usual variables, connectives, quantifiers, equality, and two one-place relations, P and Q. $P(x)$ is supposed to mean "x is a pidgeon" and $Q(x)$ is supposed to mean "x is a quibbler".

1. Translate the four sentences

All pidgeons are quibblers.
Some pidgeons are quibblers.
No pidgeons are quibblers.
Some pidgeons are not quibblers.

into logically equivalent formulas of the first-order language described above. *[4]*

NOTE: The four sentences reflect the four main sentence forms involving quantifiers studied in Aristotle's logic.

SOLUTION. Here we go:

All pidgeons are quibblers. – $\forall x\,(P(x) \to Q(x))$
Some pidgeons are quibblers. – $\exists x\,(P(x) \land Q(x))$
No pidgeons are quibblers. – $\forall x\,(P(x) \to (\neg Q(x)))$, or $(\neg \exists x\,(P(x) \land Q(x)))$
Some pidgeons are not quibblers. – $\exists x\,(P(x) \land (\neg Q(x)))$, or $(\neg \forall x\,(P(x) \to Q(x)))$

Any formulas logically equivalent to those above would do as well, of course. ■

2. If there are no pidgeons, but there are some quibblers, in the universe the four sentences in question **1** are talking about, which of them must be true? If, instead, there are no quibblers, but there are some pidgeons in that universe, which of the four sentences must be true? (Do, please, explain why in each case.) *[3]*

SOLUTION. If there are no pidgeons, but there are some quibblers, then $P(x)$ is always false and $Q(x)$ is at least sometimes true. Then:

Since $P(x)$ is always false, $P(x) \to Q(x)$ must always be true, so $\forall x\,(P(x) \to Q(x))$ is true.
Since $P(x)$ is always false, $(P(x) \land Q(x))$ is always false, so $\exists x\,(P(x) \land Q(x))$ is false.
Since $P(x)$ is always false, $P(x) \to (\neg Q(x))$ must always be true, so $\forall x\,(P(x) \to (\neg Q(x)))$ is true.
Since $P(x)$ is always false, $(P(x) \land (\neg Q(x)))$ is always false, so $\exists x\,(P(x) \land (\neg Q(x)))$ is false.

On the other hand, if there are some pidgeons, but there are no quibblers, then $P(x)$ is at least sometimes true and $Q(x)$ is always false. then:

Since $P(x)$ is true and $Q(x)$ is false for at least one value of x, $(P(x) \to Q(x))$ is false for at least one value of x, so $\forall x\,(P(x) \to Q(x))$ is false.
Since $Q(x)$ is always false, $(P(x) \land Q(x))$ is always false, so $\exists x\,(P(x) \land Q(x))$ is false.
Since $Q(x)$ is always false, $(\neg Q(x))$ is always true, so $(P(x) \to (\neg Q(x)))$ is always true, so $\forall x\,(P(x) \to (\neg Q(x)))$ is true.

Since $Q(x)$ is always false, $(\neg Q(x))$ is always true, and since $P(x)$ is sometimes true, there is at least one value of x so that $(P(x) \wedge (\neg Q(x)))$ is true, so $\exists x\, (P(x) \wedge (\neg Q(x)))$ is true.

Whew! ■

3. Aristotelian logic pretty much consists of propositional logic, plus just enough first-order logic to properly handle the four sentence forms given above. Give an example of a mathematical result and its proof that Aristotelian logic is *not* adequate to handle, and explain why this is so. *[3]*

SOLUTION. Pretty much any result that involves a statement or definition with multiple mixed quantifiers and/or binary relations will do here, because Aristotelian logic is not adapted to handling either situation. For example, consider the statement "For all real numbers a and b, if $a < b$, then there is a real number c such that $c^2 - b - a$" Have fun trying to pick the quantifiers apart using the forms given in question **1**, or trying to express $a < b$ using one-place relations if neither a or b is fixed ... ■