Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2016

Solutions to Assignment #1
Pair Constrained Triples

A certain kindergarten class has fifteen children and a slightly eccentric teacher. Each Monday the class goes for a walk to a local playground. For each such excursion, the teacher organizes the students into groups of three, and tries to ensure that no two children are in a group of three if they were together in some group of three in a previous week.

1. How many weeks can the teacher go before some pair of children have to end up together in a group of three for the second time? Provide as complete an explanation of your reasoning as you can! *[5]*

For example, with only three children in the class, it's pretty obvious that one can only go one week. With nine children, let's call them A through I, one could go four weeks:

Week 1	Week 2	Week 3	Week 4
ABC	ADG	AEI	AFH
DEF	BEH	BFG	BDI
GHI	CFI	CDH	CEG

The problem of explaining why one couldn't go a fifth week with a nine-children class is left to the reader.*

SOLUTION. The teacher can go seven weeks and no more. First, with fifteen children, call them A through O, one can go seven weeks:

Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7
AFK	ABE	BCF	EFI	GIO	IKB	KMD
BGL	CDG	DEH	GHK	HJA	JLC	LNE
CHM	HIL	IJM	LMA	CEK	EGM	OBH
DIN	JKN	KLO	NOC	DFL	FHN	ACI
EJO	MOF	NAG	BDJ	MNB	OAD	FGJ

Second, one cannot go more than seven weeks with fifteen children. Consider student A. There are fourteen other students, and hence at most seven ($7 = 14/2$) non-overlapping pairs of them that can be used to make a triple with A on successive trips before at least one of the fourteen must repeat being in the same triple as A. $\square$

NOTE: This solution – and the original problem – were taken from *Mathematical Recreations and Essays* (Second Edition) by W.W. Rouse Ball, MacMillan, London, 1905. A free electronic copy of this book can be found at: `www.gutenberg.org/ebooks/26839`

* Leaving something "to the reader" is a technique best left to professionals. :-) Please don't use it in this class . . .

2. How might you generalize and abstract this problem? What can you say about the solutions to your generalized problems? *[5]*

REMINDER: You are allowed, unless stated otherwise on an assignment, to look things up and work together. Do please provide references for any sources you end up using and make sure to write up your solutions separately.

SOLUTION. The most obvious way to generalize the problem is to consider some more-or-less arbitrary number n of children. If they are to be partitioned into triples, n will have to be a multiple of three, and reasoning similar to that given in the solution to question **1** above shows that one could go at most $(n-1)/2$ weeks without repetition. The question of whether some arrangment actually achieves $(n-1)/2$ weeks without repetition is much harder to answer, but it turns out that such an arrangement will exist if $n = 6k+3$ for some integer k.

Observing that the total number of triples in the generalized problem above will be $(n-1)/6$ for any arrangement that actually achieves $(n-1)/2$ weeks, another way to generalize the problem is simply to ask for which n there are $(n-1)/6$ possible triples so that any two children define an unique triple. A little reflection will show that n must be of the form $6k+3$ (already considered above) or $6k+1$. It turns out that suitable families of triples exist in the latter case, too.

For more on the above generalizations, read the Wikipedia article *Steiner system*. Note the connections to finite geometries ...

One could further loosen the requirements by considering maximal families of m-tuples such that any k children uniquely define an m-tuple, which gets us into the wonderful topic of *block designs* (which see), of which Steiner systems are a special case. These come in various parts of combinatorics, and have a variety of practical uses, including applications to experimental design. ■