Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2016

Take-Home Final Examination

With problem **11** *corrected!*

Due on Friday, 16 December, 2016.

Instructions: Do both of parts **N** and **Z**, and, if you wish, part **R** as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part Natural. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Define the logical connective $\downarrow$ via the following truth table:

A	B	$A \downarrow B$
T	T	F
T	F	F
F	T	F
F	F	T

Show how to write formulas equivalent to each of $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$, and $A \leftrightarrow B$ using just the connective $\downarrow$ (you may, of course, use it more than once in each formula), or explain in each case that it can't be done why it is so.

2. Suppose that $k = a^2 + b^2$ and $n = c^2 + d^2$ for some a, b, c, $d \in \mathbb{N}$. Show that $kn = e^2 + f^2$ for some e, $f \in \mathbb{N}$. (You may assume that $+$ and $\cdot$ have all the usual algebraic properties on $\mathbb{N}$.)

3. Suppose $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences of rational numbers, and define the sequence $\{c_n\}$ by setting $c_n = a_n^2 - b_n^2$ for each $n \in \mathbb{N}$. Show that $\{c_n\}$ is also a Cauchy sequence.

4. In a certain mathematics class Professor B, who always tells the truth and is never mistaken*, explains the marking scheme for the course to the students.

> "This course meets once each week. There will be only one test, which will be written in class in one of the next twelve weeks. However, you will not know which week it is until the class in which the test is given."

Is there any way to determine in which week the test is given? Explain why or why not. If there is, in which week will the test be written?

More exam on page 2 . . .

* Yes, please *do* suspend your disbelief! :-)

Part Zahl. Do any *four* (4) of problems **5** – **11**. *[40 = 4 × 10 each]*

5. Suppose $P(x)$ is a statement in a first-order language. Write a formula in the language that expresses the statement "There are exactly three possible values of x for which $P(x)$ is true."

6. Let $\mathbb{I} = \{ a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z} \}$. Determine whether $\mathbb{I}$ is countable or not.

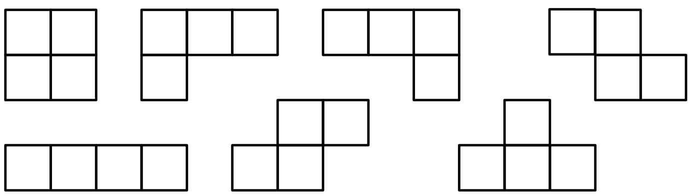
7. Suppose $n > 1$ is a natural number such that $p = 3 \cdot 2^{n-1} - 1$, $q = 3 \cdot 2^n - 1$, and $r = 9 \cdot 2^{2n-1} - 1$ are all prime numbers. Show that $a = p \cdot q \cdot 2^n$ and $b = r \cdot 2^n$ are a pair of *amicable numbers*, that is, each is the sum of the other's divisors (not including the other itself).

8. Suppose the schnitt A represents the positive real number r. Use A to define the schnitt representing the real number $\frac{1}{r}$.

9. Suppose $\{a_n\}$ is a sequence such that $\lim_{n \to \infty} a_n = L$. Let $b_n = \inf \{ a_k \mid k \geq n \}$ for each $n \geq 0$. Show that $\lim_{n \to \infty} b_n = L$ too.

10. Suppose p is a prime number. Show that $\sqrt{p}$ is irrational.

11. *Tetrominoes* are shapes obtained by glueing four 1×1 squares together full edge to full edge. In some cases, such as the game *Tetris*, two tetrominoes that can be made congruent via rotations are considered to be the same, but reflections (*i.e.* flips) are not allowed. This gives ~~five~~ seven different tetrominoes:



a. Show how to completely cover an 8×8 square with non-overlapping tetrominoes, using each tetromino at least once and without having any extend beyond the 8×8 square, or explain why no such covering can exist. *[5]*

b. Show how to completely cover a 9×10 rectangle with non-overlapping tetrominoes, using each tetromino at least once and without having any extend beyond the 9×10 rectangle, or explain why no such covering can exist. *[5]*

[Total = 80]

Part Real! Bonus ...

***ρ*.** Write an original poem about logic or mathematics. *[1]*

I HOPE THAT YOU ENJOYED THIS COURSE. HAVE AN EVEN BETTER BREAK!