Mathematics 2200H – Mathematical Reasoning

TRENT UNIVERSITY, Fall 2016

Assignment #8

Suprema and Infima

Due on Thursday, 10 November.

The *supremum* or *least upper bound* of a non-empty set X of real numbers is the real number a such that:

- a is an upper bound for X, *i.e.* $x \leq a$ for all $x \in X$.
- If u is any upper bound for X, then $a \leq u$.

Similarly, the *infimum* or *greatest lower bound* of a non-empty set X of real numbers is the real number b such that:

- b is a lower bound for X, *i.e.* $b \leq x$ for all $x \in X$.
- If ℓ is any lower bound for X, then $\ell \leq b$.

Note that X can have a supremum, usually denoted by $\sup(X)$, only if it is bounded above, and an infimum, usually denoted by $\inf(X)$, only if it is bounded below.

1. Using the definition of real numbers as schnitts, show that $\sup(X) = \bigcup_{x \in X} x$ whenever $X \neq \emptyset$ is a set of real numbers with an upper bound. Can you find a similar way to obtain $\inf(X)$ in terms of schnitts whenever $X \neq \emptyset$ is a set of real numbers with a lower bound? *[4]*

2. Suppose $\emptyset \neq X \subset \mathbb{R}$ is bounded above, and let $Y = \{\, -x \mid x \in X \,\}$. Show that Y is bounded below and that $\inf(Y) = -\sup(X)$. *[3]*

3. Suppose $\emptyset \neq Y \subseteq X \subset \mathbb{R}$ and X (and hence also Y) is bounded above and below. Show that $\inf(X) \leq \inf(Y) \leq \sup(Y) \leq \sup(X)$. *[2]*

4. Suppose $\emptyset \neq X \subset \mathbb{R}$ is a set such that $\inf(X) = \sup(X)$. What can you deduce about X? *[1]*