Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2016

Assignment #7
The reals defined via an equivalence relation
Due on Thursday, 3 November.

We will look at how to define the real numbers from the rational numbers using *Dedekind cuts*, also known as *schnits*, in class. In this assignment we will consider an alternate way of defining the real numbers using equivalence classes of sequences of rational numbers. We will assume in what follows that we have the rational numbers, the usual operations and relations on them, and their properties in hand.

> DEFINITION. A sequence $\langle a_n \rangle = \{a_n \mid n \in \mathbb{N}\}$ of rational numbers is a *Cauchy sequence* if, for all rational numbers $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all natural numbers n, $m \geq N$, $|a_n - a_m| < \varepsilon$.

That is, a sequence is a Cauchy sequence if it ought to converge to some (real!) number; of course, some of the potential limits, such as $\sqrt{2}$, are not to be found in $\mathbb{Q}$.

1. Give an example of a sequence $\langle a_n \rangle$ of rational numbers for which for all rational numbers $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all natural numbers $n \geq N$, $|a_n - a_{n+1}| < \varepsilon$, but which is *not* a Cauchy sequence. *[3]*

Let $\mathcal{C}$ denote the set of all Cauchy sequences of rational numbers; that is, $\mathcal{C} = \{ \langle a_n \rangle \mid \langle a_n \rangle \text{ is a Cauchy sequence of rationals} \}$.

> DEFINITION. The binary relation $\equiv$ on $\mathbb{C}$ is defined by setting $\langle a_n \rangle \equiv \langle b_n \rangle$ if, and only if, for all rational numbers $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all natural numbers $n \geq N$, $|a_n - b_n| < \varepsilon$.

2. Verify that $\equiv$ is an equivalence relation on $\mathcal{C}$. *[3]*

Given that $\equiv$ is indeed an equivalence relation on $\mathcal{C}$, then the *equivalence class* of $\langle a_n \rangle$ is the set $[\langle a_n \rangle]_\equiv = \{ \langle b_n \rangle \in (C) \mid \langle a_n \rangle \equiv \langle b_n \rangle \}$, and we can now define the set of real numbers to be $\mathbb{R} = \{ [\langle a_n \rangle]_\equiv \mid \langle a_n \rangle \in \mathcal{C} \}$.

3. Define $+_\mathbb{R}$ and $\cdot_\mathbb{R}$, the operations of addition and multiplication on the real numbers (defined as above), and show that they are well-defined. *[4]*