Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2016

Assignment #6
The integers defined via an equivalence relation
Due on Thursday, 20 October.

Here is the way, mentioned in class, of defining the integers from the natural numbers that is analogous to the way the rationals were defined in class from the integers.

Let $\mathbb{N}\times\mathbb{N} = \{\, (a,b) \mid a,\, b \in \mathbb{N}\,\}$ be the collection of all ordered pairs of natural numbers. Define the binary relation $\sim$ on $\mathbb{N}\times\mathbb{N}$ by setting $(a,b) \sim (c,d)$ if and only if $a+d = b+c$, where $+$ is the usual operation of addition on the natural numbers. Intuitively, $(a,b) \sim (c,d)$ exactly when $a-b = c-d$.

1. Verify that $\sim$ is an *equivalence relation* on $\mathbb{N}\times\mathbb{N}$. *[4]*

That is, you need to check the following:

i. $\sim$ is *reflexive*: $(a,b) \sim (a,b)$ for all $(a,b) \in \mathbb{N}\times\mathbb{N}$.

ii. $\sim$ is *commutative*: $(a,b) \sim (c,d)$ if and only if $(c,d) \sim (a,b)$ for all (a,b), $(c,d) \in \mathbb{N}\times\mathbb{N}$.

iii. $\sim$ is *transitive*: $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$ imply that $(a,b) \sim (e,f)$ for all (a,b), (c,d), $(e,f) \in \mathbb{N}\times\mathbb{N}$.

Given that $\sim$ is indeed an equivalence relation on $\mathbb{N}\times\mathbb{N}$, then the *equivalence class* of (a,b) is the set $[(a,b)]_\sim = \{\, (c,d) \in \mathbb{N}\times\mathbb{N} \mid (a,b) \sim (c,d)\,\}$, and we now define the set of integers to be $\mathbb{Z} = \{\, [(a,b)]_\sim \mid (a,b) \in \mathbb{N}\times\mathbb{N}\,\}$.

2. Define $+_\mathbb{Z}$, the operation of addition on the integers (defined as above), and show that it is associative. *[3]*

3. Define $\cdot_\mathbb{Z}$, the operation of multiplication on the integers (defined as above), and show that it is commutative. *[3]*

In all of the above problems, you may assume that $+$ and $\cdot$ have been defined on $\mathbb{N}$, and have the usual algebraic properties.