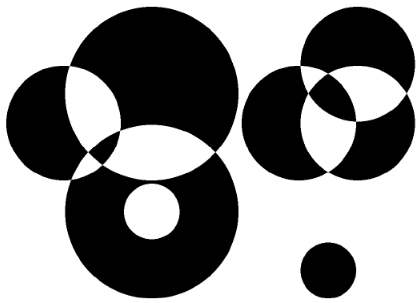
Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2015
Take-Home Final Examination
Due on Friday, 18 December, 2015.

Instructions: Do both of parts **P** and **Q**, and, if you wish, part **R** as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part Proposition. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Suppose finitely many, possibly overlapping, circles are drawn in the plane, dividing it into regions whose borders are made up of circular arcs. Show that the regions can be coloured using white and black so that no two regions sharing a common border have the same colour.



***β*.** *Bonus!* Draw a configuration of circles and a colouring of the regions they create that demonstrates this is a Mickey Mouse problem. *[0.5]*

2. Recall that the inhabitants of the Island of Knights and Knaves are either knights, who always tell the truth, or knaves, who always lie. While visiting the Island you encounter . . .

a. . . . two inhabitants, A and B, who tell you the following:
A: If B is a knave then I'm not a knight.
B: One of us is a knight and the other is a knave.
Determine, as completely you can, whether each of A and B is a knight or a knave. *[3]*

b. . . . seven inhabitants, A, B, C, D, E, F, and G, who tell you the following:
A: B is a knave or C is a knight.
B: B and E are both knights or both knaves.
C: B and F are both knights.
D: A and C are both knaves.
E: C is a knave.
F: Exactly one of B and G is a knight.
G: One of A and B is a knave and the other is a knight.
Determine, as completely as you can, whether each of A, B, C, D, E, F, and G is a knight or a knave. *[7]*

More exam on page 2 . . .

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ x^3 & x \notin \mathbb{Q} \end{cases}$. Use the ε-δ or the sequential definition of limits of functions, as appropriate, to verify that:

a. $\lim_{x \to 0} f(x) = 0$ *[3]* **b.** $\lim_{x \to 1} f(x) = 1$ *[3]* **c.** $\lim_{x \to 2} f(x)$ does not exist. *[4]*

4. Suppose a and b are schnitts. Determine whether $D = \{\, s - t \mid s \in a \wedge t \in b \,\}$ is a schnitt. If it isn't, check to see which parts of the definition of schnitt D does satisfy.

Part Quantifier. Do any *four* (4) of problems **5** – **11**. *[40 = 4 × 10 each]*

5. Define the logical connective $\uparrow$ via the following truth table:

A	B	$A \uparrow B$
T	T	F
T	F	T
F	T	T
F	F	T

Show how to write formulas equivalent to $\neg A$, $A \vee B$, $A \wedge B$, $A \Rightarrow B$, and $A \Leftrightarrow B$ using just the connective $\uparrow$. (You may use it more than once in each formula, of course.)

6. Suppose $\{a_n\}$ is a sequence with $\lim_{n \to \infty} a_n = L$. Let $b_n = \sup \{\, a_k \mid k \geq n \,\}$ for each $n \geq 0$. Show that $\lim_{n \to \infty} b_n = L$ too.

7. $\{f_n\}$ is defined inductively by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$. Use induction to show that for all $n \geq 1$, $f_n = \dfrac{1}{\sqrt{5}} \left[\left(\dfrac{1 + \sqrt{5}}{2} \right)^n - \left(\dfrac{-2}{1 + \sqrt{5}} \right)^n \right]$.

8. Suppose $A \neq \emptyset$ is a set and $f : A \to A$ is a 1-1 function with $B = \{\, f(a) \mid a \in A \,\} \neq A$. Show that A cannot be finite, *i.e.* it does not have n elements for some $n \in \mathbb{N}$.

9. Suppose $n \in \mathbb{N}$ and $n \geq 2$. Show that:

a. $3^n - 1$ is not prime. *[1]*

b. If n is not prime, then neither is $2^n - 1$. *[9]*

10. Let $\mathbb{I} = \{\, a + bi + cj + dk \in \mathbb{H} \mid a, b, c, d \in \mathbb{Z} \,\}$. Determine whether $\mathbb{I}$ is countable.

11. *Pentominoes* are shapes obtained by gluing five 1×1 squares together full edge to full edge. Two pentominoes that can be made congruent via reflections (*i.e.* flips) or rotations are considered to be the same. Find all twelve pentominoes and an arrangement of all of them into a 6×10 rectangle.

[Total = 80]

Part Rhyme! More bonus …

α. Write an original poem about logic or mathematics. *[1]*

I HOPE THAT YOU ENJOYED THIS COURSE. HAVE A GOOD BREAK!