Mathematics 2200H – Mathematical Reasoning

TRENT UNIVERSITY, Fall 2015

Assignment #7

Due on Tuesday, 10 November, 2015.

Quaternions

Recall from class that the *quaternions* are the number system

$$\mathbb{H} = \{\, a + bi + cj + dk \mid a,\, b,\, c,\, d \in \mathbb{R} \,\}$$

where $+$ and $\cdot$ work as usual except for the special numbers i, j, and k, which satisfy the following equations:

$$\begin{array}{lll} i^2 = j^2 = k^2 = -1 & & \\ ij = k & jk = i & ki = j \\ ji = -k & kj = -i & ik = -j \end{array}$$

(That is, the quaternions are like the complex numbers, only more so. :-) Note that multiplication is not always commutative in the quaternions.

1. Suppose $h = a + bi + cj + dk \neq 0$ is a quaternion. Find $h^{-1} = \frac{1}{h}$. *[3]*

2. Suppose $\mathbf{a} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} t \\ u \\ v \end{bmatrix}$ are vectors in $\mathbb{R}^3$, and that $(pi + qj + rk)(ti + uj + vk) = a + bi + cj + dk$ in $\mathbb{H}$. Verify that $\mathbf{a} \times \mathbf{b} = \begin{bmatrix} b \\ c \\ d \end{bmatrix}$. What does a represent in terms of $\mathbf{a}$ and $\mathbf{b}$? *[3]*

3. Find 4×4 matrices $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ with real entries such that $\mathbf{U}^2 = \mathbf{V}^2 = \mathbf{W}^2 = -\mathbf{I}_4$, $\mathbf{UV} = \mathbf{W}$, $\mathbf{VW} = \mathbf{U}$, $\mathbf{WU} = \mathbf{V}$, $\mathbf{VU} = -\mathbf{W}$, $\mathbf{WV} = -\mathbf{U}$, and $\mathbf{UW} = -\mathbf{V}$. *[4]*

NOTE: It follows from this that you can find a copy of the quaternions among the 4×4 real matrices.