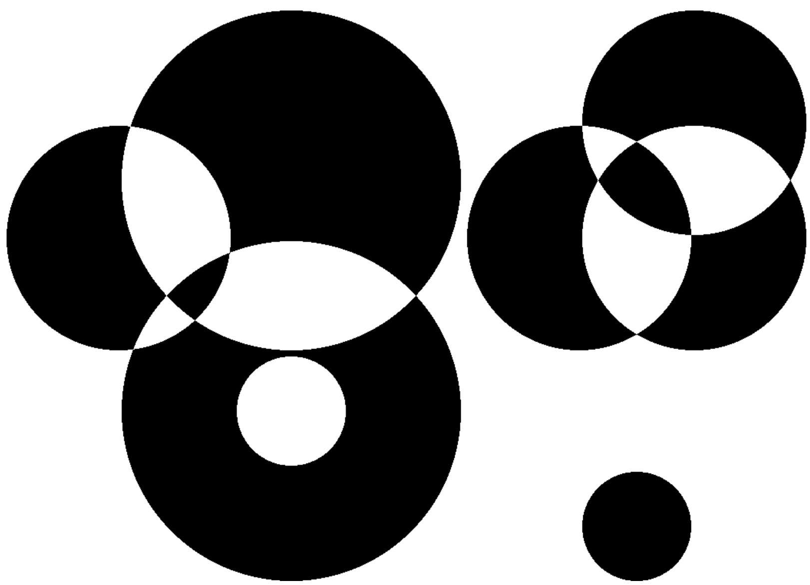
Mathematics 2084H – Recreational mathematics
TRENT UNIVERSITY, Winter 2009

TAKE-HOME FINAL EXAMINATION
Due on Thursday, 23 April, 2009.

Instructions: Give complete answers to receive full credit, including references to any and all sources you used. You may use your texts from this and any other courses, as well as any handouts, class notes, and the like; you may also ask the instructor to clarify the instructions or any of the questions; and you may use a calculator or computer to perform any necessary calculations. *You may not consult any other sources, nor give or receive any other aid on this exam, except with the intructor's explicit permission.*

Part I – This . . . Do any *four* of **1** – **7**. *[40 = 4 × 10 each]*

1. Suppose a number of circles, which may overlap, are drawn on a blank piece of paper, dividing up the paper into a number of regions whose borders are made up of circular arcs. One can colour these regions with only two colours in so that no two regions that have a length of border in common have the same colour. An example is given below.



Explain why one can always colour regions created in such a way with only two colours so that no two regions that have a common border have the same colour. *[10]*

2. Find a dissection from a domino (that is, a rectangle with proportions 2 : 1) to a square. *[10]*

3. Find a shape other than a hexagon which can only tile the plane periodically. *[10]*

4. Find a $4 \times 4 \times 4$ *Latin cube*: a three-dimensional array using each of the numbers 1, 2, 3, and and 4 exactly 16 times so that each of these numbers occurs exactly once in each (sideways) row, (vertical) column, and (back-to-front) file. Alternatively, explain why such a cube cannot exist. *[10]*

5. Recall that *tetrominoes* are the shapes obtained by attaching four squares of equal size together edge-to-edge.

a. Find all the possible tetrominoes if tetrominoes that can be made congruent via reflections (*i.e.* flips) and rotations are considered to be the same. *[2]*

b. Find all the possible tetrominoes if only tetrominoes that can be made congruent via rotations are considered to be the same*. *[2]*

c. Find a tiling of an 8×8 chessboard using each of the distinct tetrominoes – as defined in **a** – the same number of times, or explain why there is no such tiling. *[3]*

d. Find a tiling of an 8×8 chessboard using each of the distinct tetrominoes – as defined in **b** – at least once, or explain why there is no such tiling. *[3]*

6. An $n \times n$ array of integers is *semimagic* if the entries in every row or column, but not necessarily the diagonals, add up to he same constant.

a. Explain how to use a pair of mutually orthogonal $n \times n$ latin squares to construct an $n \times n$ semimagic square. *[5]*

b. Explain how to use a $n \times n$ semimagic square that uses the numbers 1 through n^2 to construct a pair of mutually orthogonal $n \times n$ latin squares. *[5]*

7. Recall that every native of the Island of Knights and Knaves is either a knight who always tells the truth or a knave who always lies. You meet six natives, who tell you the following:

A: Both D and E are knaves.
B: E is a knave or F is a knave.
C: E is a knight and F is a knight.
D: C could say that B is a knight.
E: D is a knave.
F: C could say that B is a knave.

Determine, as best you can, which of A–F are knights and which are knaves†. *[10]*

* As in Tetris!

† This puzzle was adapted from one by Zachary Ernst

Part II – ... and that! Do any *two* of **8–11**. *[20 = 2 × 10 each]*

8. *Brussel sprouts* is a relative of the game sprouts discussed in class:

> Brussels sprouts begins with n crosses instead of spots. A move consists of extending any arm of any cross into a curve that ends at the free arm of any other cross or the same cross; then a crossbar is drawn anywhere along the curve to create a new cross. Two arms of the new cross will, of course, be dead, since no arm may be used twice. As in sprouts, no curve may cross itself or cross a previously drawn curve, nor may it go through a previously made cross. As in sprouts, the winner of the normal game is the last person to play and the winner of the misère game is the first person who cannot play.‡

a. Show that every game of Brussel sprouts starting with n crosses must end in exactly $5n - 2$ moves. *[5]*

b. Given **a**, what is the optimal strategy for Brussel sprouts? *[5]*

9. Consider the following modification of *Division Nim*:

> Two players, A and B, each write down a number greater than 1, which serve as the starting numbers for the game. In turn each player divides either one (but not both) of the two numbers available at the start of their turn by one of its divisors (excluding '1'). The first player to reach '1' for both numbers is the loser. Thus a game might run as follows:

A chooses 21 and B chooses 18.

Starting numbers:	21	18
A divides 18 by 3	21	6
B divides 6 by 3	21	2
A divides 21 by 3	7	2
B divides 2 by 2	7	1
A divides 7 by 7	1	1
	(... so A loses)	

a. Assuming that A chooses 21 and B chooses 18, as in the example above, who should win if both players play optimally? *[4]*

b. In general, if A chooses $n > 1$ and B chooses $k > 1$ to start with, what is the optimal strategy for the two players to pursue? *[6] [10]*

‡ This description was taken from Martin Gardner's article *Sprouts and Brussel Sprouts*, which is reprinted in the collection *Mathematical Carnival*, by Martin Gardner, Mathematical Association of America, Washington, 1989, ISBN 0-88385-448-1.

10. Does the sudoku puzzle

								7
				7				
		6						
		4						
			1	5				8
					2		7	
					1	4		
					4			
1								

have no solution, just one solution, or multiple solutions? *[10]*

11. Recall that *pentominoes* are the shapes obtained by attaching five squares of equal size together edge-to-edge and that two pentominoes that can be made congruent via reflections (*i.e.* flips) and rotations are considered to be the same. Find an arrangement of the twelve pentominoes into a 5×12 rectangle and distribute the integers 1 through 60 among the 60 squares of the twelve pentominoes so as to satisfy as many as you can of the following conditions:

i. the sum of the squares of each pentomino is the same,
ii. each row of the rectangle has the same sum as every other row, and
iii. each column of the rectangle has the same sum as every other column. *[10]*

[Total = 60]

Part III - Just one more thing! Bonus!

◯. Write an orignal poem touching on receational mathematics. *[1]*

I HOPE YOU ENJOYED THE COURSE! HAVE A GOOD SUMMER!