MATH 1550
Fall 2023
Test 2
2023-3-15
Time Limit: 50 minutes

Name (Print): ____________________

ID number: ____________________

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator, so long as it is not capable of wirelessly communicating with other devices or accessing the internet.
- A single leaf, double-sided, A4 size, aid sheet is allowed.

1. (5 points) Do any 2 of the following for 2.5 points each:

   (a) Suppose $F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 0 \end{cases}$ is the cumulative distribution function of the continuous random variable X. What is the probability density function of X?

   (b) A fair four-sided die has faces numbered 1, 2, 3, and 5. What is the expected value of the number that comes up when the die is rolled once?

   (c) If $f(t) = \begin{cases} 1+t & t \in [-1,0] \\ 1-t & t \in [0,1] \\ 0 & \text{otherwise} \end{cases}$ is the probability density function of the continuous random variable T. Find $P(-0.5 \leq T \leq 0.5)$.

solution:

(a) The probability density function $f(x)$ is the derivative of the cumulative distribution function. Since $\frac{d}{dx}0 = 0$, $\frac{d}{dx}x = 1$, and $\frac{d}{dx}1 = 0$, it follows that $f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

(b) Suppose the random variable X gives the number on the face that comes up when the die is rolled. Since the die is fair, all four faces have an equal chance to come up, so $P(X = 1) = P(X = 2) = P(X = 3) = P(X = 5) = \frac{1}{4}$. Applying the definition of expected value, we get:

$$\begin{aligned} E(X) &= \sum_x xP(X = x) = 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + 3 \cdot P(X = 3) + 5 \cdot P(X = 5) \\ &= 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 5 \cdot \frac{1}{4} = \frac{11}{4} = 2.75 \end{aligned}$$

(c) By definition,

$$\begin{aligned} P(-0.5 \leq T \leq 0.5) &= \int_{-0.5}^{0.5} f(t)\,dt = \int_{-0.5}^{0} (1+t)\,dt + \int_{0}^{0.5} (1-t)\,dt \\ &= \left(t + \frac{t^2}{2}\right)\Big|_{-0.5}^{0} + \left(t - \frac{t^2}{2}\right)\Big|_{0}^{0.5} \\ &= \left(0 + \frac{0^2}{2}\right) - \left(-0.5 + \frac{(-0.5)^2}{2}\right) + \left(0.5 - \frac{(0.5)^2}{2}\right) - \left(0 - \frac{0^2}{2}\right) \\ &= 0 - (-0.5 + 0.125) + (0.5 - 0.125) - 0 = -(-0.375) + 0.375 \\ &= 0.75 = \frac{3}{4}. \end{aligned}$$

□

2. (5 points) Do any 2 of the following for 2.5 points each:

   (a) Determine whether $f(x,y) = \begin{cases} x^2y^2 & x \in [-1,1] \text{ and } y \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$, is a valid joint probability density function.

   (b) The joint probability distribution function $f(x,y)$ of the discrete random variables X and Y is given by the table below. Find $P(X = 1|Y = 5)$.

y \ x	0	1	2
3	0.1	0	0.1
4	0.1	0.1	0.2
5	0.2	0.1	0.1

   (c) Two fair six-sided dice are given, one blue and one red. The blue one has faces numbered 0 through 5, and the red one has faces numbered 2 through 7. One of these dice is selected at random (with equal probability) and then rolled once. The random variable X counts the number of times the blue die is chosen, and the random variable Y gives the number on the face that comes up when the selected die is rolled. Give the complete table describing the joint distribution function of X and Y.

solution:

(a) Note that $f(x,y) \geq 0$ for all x and y. Now,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} x^2y^2\,dxdy + \int\int_{\text{otherwise}} 0\,dxdy$$
$$= \int_{-1}^{1} y^2 \cdot \left.\frac{x^3}{3}\right|_{x=-1}^{x=1} dy = \int_{-1}^{1} y^2\left(\frac{1^3}{3} - \frac{(-1)^3}{3}\right) dy$$
$$= \int_{-1}^{1} \frac{2}{3}y^2\,dy = \frac{2}{3}\cdot\left.\frac{y^3}{3}\right|_{-1}^{1} = \frac{2}{3}\left(\frac{1^3}{3} - \frac{(-1)^3}{3}\right) = \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9} \neq 1$$

Since $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy \neq 1$, $f(x,y)$ fails to be a valid probability density.

(b) We use the definition of conditional probability and compute away:

$$P(X = 1|Y = 5) = \frac{P(X = 1 \text{ \& } Y = 5)}{P(Y = 5)} = \frac{f(1,5}{f(0,5) + f(1,5) + f(2,5)}$$
$$= \frac{0.1}{0.2 + 0.1 + 0.1} = \frac{0.1}{0.4} = \frac{1}{4} = 0.25$$

(c)

x \ y	0	1	2	3	4	5	6	7
0	0	0	1/12	1/12	1/12	1/12	1/12	1/12
1	1/12	1/12	1/12	1/12	1/12	1/12	0	0

□

3. (5 points) Do any 1 of the following for 5 points:

   (a) Let $f(x) = \begin{cases} 4\left(x^3 - x\right) & -1 \leq x \leq 0 \\ 0 & \text{otherwise} \end{cases}$, be the probability density function of the continuous random variable X. Find the expected value, $E(X)$, of X.

   (b) The joint probability distribution function $f(x, y)$ of the discrete random variables X and Y is given by the table below. Find the expected value, $E(Y - X)$, of $Y - X$.

		x = 0	x = 1	x = 2
y	3	0.1	0	0.1
	4	0.1	0.1	0.2
	5	0.2	0.1	0.1

solution:

(a) Here goes:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x)\,dx = \int_{\infty}^{-1} x \cdot 0\,dx + \int_{-1}^{0} x \cdot 4\left(x^3 - x\right)\,dx + \int_{0}^{\infty} x \cdot 0\,dx \\ &= \int_{\infty}^{-1} 0\,dx + 4\int_{-1}^{0} \left(x^4 - x^2\right)\,dx + \int_{0}^{\infty} 0\,dx = 0 + 4\left(\frac{x^5}{5} - \frac{x^3}{3}\right)\Big|_{-1}^{0} + 0 \\ &= 4\left(\frac{0^5}{5} - \frac{0^3}{3}\right) - 4\left(\frac{(-1)^5}{5} - \frac{(-1)^3}{3}\right) = 0 - 4\left(\frac{-1}{5} - \frac{-1}{3}\right) = -4\left(-\frac{3}{15} + \frac{5}{15}\right) \\ &= -4 \cdot \frac{2}{15} = -\frac{8}{15} \approx -0.5333 \end{aligned}$$

(b) We apply the definition of expected value in this case and do a lot of arithmetic:

$$\begin{aligned} E(Y - X) &= \sum_{y} \sum_{x} (y - x) f(x, y) \\ &= (3-0) \cdot 0.1 + (3-1) \cdot 0 + (3-2) \cdot 0.1 \\ &\quad + (4-0) \cdot 0.1 + (4-1) \cdot 0.1 + (4-2) \cdot 0.2 \\ &\quad + (5-0) \cdot 0.2 + (5-1) \cdot 0.1 + (5-2) \cdot 0.1 \\ &= 0.3 + 0 + 0.1 + 0.4 + 0.3 + 0.4 + 1.0 + 0.4 + 0.3 = 3.2 \end{aligned}$$

□

BonusBonusBonus

4. (1 Bonus points) One hundred people line up to board an airplane. Each has a boarding pass with assigned seat. However, the first person to board has lost their boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of the unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in their assigned seat?

solution:

Look at the situation when the kth passenger enters. Neither of the previous passengers showed any preference for the kth seat *vs.* the seat of the first passenger. This in particular is true when $k = n$. But the nth passenger can only occupy their seat or the first passenger's seat. Therefore the probability is $1/2$.

$\square$