MATH 1550
Fall 2023
Test 1
2023-2-8
Time Limit: 50 minutes

Name (Print): ____________________

ID number: ____________________

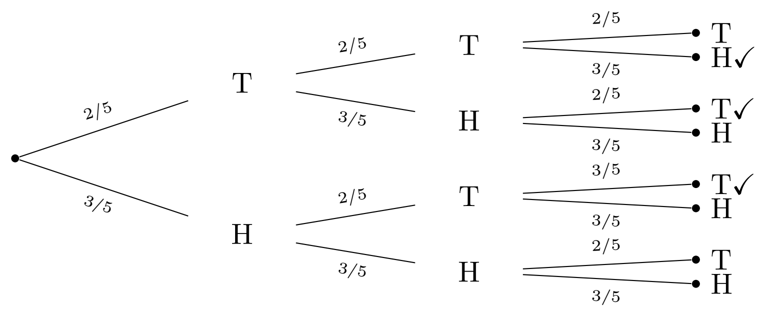
Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator, so long as it is not capable of wirelessly communicating with other devices or accessing the internet.

1. (5 points) Do any 2 of the following for 2.5 points each:
   (a) A biased coin with $P(H) = \frac{3}{5}$ and $P(T) = \frac{2}{5}$ is tossed three times. Use a tree diagram to determine the probability of obtaining exactly two tails in this experiment.
   (b) A hand of five cards is drawn randomly and simultaneously from a standard 52-card deck. Ignoring order, how many possible hands have two $\heartsuit$s, two $\diamondsuit$s, and one $\spadesuit$?
   (c) Suppose that A and B are events in some sample space, with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \cup B) = \frac{2}{3}$. What is $P\left((A \cap B)'\right)$?

solution:

(a)



There are 3 different ways to obtain exactly 2 tails on the 3 tosses. Multiply the probabilities on the edges the paths to obtain each of the 3 probabilities. The total probability is therefore

$$\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)+\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)+\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)=\frac{36}{125}$$

(b) The number of such hands is

$$\binom{13}{2}\binom{13}{2}\binom{13}{1} = 79092.$$

(c)

$$P((A \cap B)') = 1 - P(A \cap B) = 1 + (P(A \cup B) - P(A) - P(B)) = 1 + \frac{2}{3} - \frac{1}{2} - \frac{1}{3} = \frac{5}{6}.$$

□

2. (5 points) Do any 2 of the following for 2.5 points each:

   (a) A fair coin is tossed three times. The random variable X computes the number of heads that come up minus the number of tails that come up in this experiment. What is the cumulative distribution function of X?

   (b) A hand of five cards is drawn randomly and simultaneously from a standard 52-card deck. Let A be the event that the hand consists of $A\heartsuit$, $K\heartsuit$, $Q\diamondsuit$, $J\diamondsuit$, and $10\spadesuit$. and let B be the event that the hand includes two $\heartsuit$'s, two $\diamondsuit$'s, and one $\spadesuit$. Compute $P(A|B)$ and $P(B|A)$.

   (c) A fair coin is tossed repeatedly until it comes up with the same face that came up on the first toss. What is the sample space for this experiment and what is the probability of each possible outcome?

solution:

(a) The range of X is $\{-3,-1,1,3\}$, and the probability distribution and cumulative distribution for X are,

x	-3	-1	1	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

and

$$P(X\le x)=\begin{cases}0 & \text{for } x<-3\\ \frac{1}{8} & \text{for } -3\le x<-1\\ \frac{4}{8} & \text{for } -1\le x<1\\ \frac{7}{8} & \text{for } 1\le x<3\\ 1 & \text{for } \ge 3\end{cases}$$

(b)

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A)}{P(B)}=\frac{\frac{1}{\binom{52}{5}}}{\frac{\binom{13}{2}\binom{13}{2}\binom{13}{1}}{\binom{52}{5}}}=\frac{1}{79092},\quad P(B|A)=1.$$

(c) The sample space for the experiment is the set

$$S\{HH,HTH,HTTH,HTTTH,\ldots,TT,THT,THHT,THHHT,\ldots\}.$$

Let H_i/T_i be the event that heads/tails is tossed first, and it takes i additional tosses to obtain heads/tails again; i.e. $H_1=\{HH\}$, $H_2=\{HTH\}$, $T_4=\{THHHT\}$, etc. Then

$$P(H_i)=\frac{1}{2^{i+1}},\quad P(T_i)=\frac{1}{2^{i+1}}.$$

□

3. (5 points) Jar 1 contains three blue and two yellow balls, and Jar 2 contains two blue and three yellow balls. A blindfolded experimenter chooses one of the jars at random and then chooses a ball from the chosen jar at random. If the chosen ball is yellow, what is the probability that Jar 2 had been chosen?

solution:

By the rule of total probability

$$P(\text{yellow}) = P(\text{Jar 1}) \cdot P(\text{yellow}|\text{Jar 1}) + P(\text{Jar 2}) \cdot P(\text{yellow}|\text{Jar 2}) = \frac{1}{2}\left(\frac{2}{5}\right) + \frac{1}{2}\left(\frac{3}{5}\right) = \frac{1}{2},$$

and by Baye's Theorem

$$P(\text{Jar 2}|\text{yellow}) = \frac{P(\text{Jar 2}) \cdot P(\text{yellow}|\text{Jar 2})}{P(\text{yellow})} = \frac{\frac{1}{2}\left(\frac{3}{5}\right)}{\frac{1}{2}} = \frac{3}{5}.$$

□

BonusBonusBonus

4. (1 Bonus points) If you were to pick an answer to this question at random from among the choices below, what is the probability that it would be correct? Justify your answer!

(a) 0.2 (b) $1/\pi$ (c) 1/5 (d) 0.0 (e) 4/10