

1. Let  $X$  and  $Y$  be discrete random variables with distributions

$$f(x) = \begin{cases} \frac{1}{3} & x = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases} \quad g(y) = \begin{cases} \frac{1}{3} & y = -2, 0, 2 \\ 0 & \text{otherwise} \end{cases},$$

respectively.

- (a) Find the mean and variance for  $X$ .
- (b) Find the mean and variance for  $Y$ .

*Solution.* (a) Mean:

$$\mu = E(X) = \sum_x xf(x) = (-1) \cdot \frac{1}{3} + (0) \cdot \frac{1}{3} + (1) \cdot \frac{1}{3} = 0.$$

Variance:

$$\sigma^2 = E((X - \mu)^2) = \sum_x (x - \mu)^2 f(x) = (-1 - 0)^2 \cdot \frac{1}{3} + (0 - 0)^2 \cdot \frac{1}{3} + (1 - 0)^2 \cdot \frac{1}{3} = \frac{2}{3},$$

or

$$\sigma^2 = E(X^2) - \mu^2 = \sum_x x^2 f(x) - \mu^2 = \left( (-1)^2 \cdot \frac{1}{3} + (0)^2 \cdot \frac{1}{3} + (1)^2 \cdot \frac{1}{3} \right) - 0^2 = \frac{2}{3}.$$

- (b) Mean:

$$\mu = E(Y) = \sum_y yg(y) = (-2) \cdot \frac{1}{3} + (0) \cdot \frac{1}{3} + (2) \cdot \frac{1}{3} = 0.$$

Variance:

$$\sigma^2 = E(Y^2) - \mu^2 = \sum_y y^2 g(y) = \left( (-2)^2 \cdot \frac{1}{3} + (0)^2 \cdot \frac{1}{3} + (2)^2 \cdot \frac{1}{3} \right) - 0^2 = \frac{8}{3}.$$

□

2. Let random variable  $X$  be sum of two regular six sided dice. Find the mean and variance of  $X$ .

*Solution.* Mean:

$$\begin{aligned} \mu &= E(X) \\ &= \sum_x xf(x) \\ &= (2) \cdot \frac{1}{36} + (3) \cdot \frac{2}{36} + (4) \cdot \frac{3}{36} + (5) \cdot \frac{4}{36} + (6) \cdot \frac{5}{36} + (7) \cdot \frac{6}{36} \\ &\quad + (8) \cdot \frac{5}{36} + (9) \cdot \frac{4}{36} + (10) \cdot \frac{3}{36} + (11) \cdot \frac{2}{36} + (12) \cdot \frac{1}{36} \\ &= 7 \end{aligned}$$

Variance:

$$\begin{aligned}
\sigma^2 &= E((X - \mu)^2) \\
&= \sum_x (x - \mu)^2 f(x) \\
&= (2 - 7)^2 \cdot \frac{1}{36} + (3 - 7)^2 \cdot \frac{2}{36} + (4 - 7)^2 \cdot \frac{3}{36} + (5 - 7)^2 \cdot \frac{4}{36} + (6 - 7)^2 \cdot \frac{5}{36} + (7 - 7)^2 \cdot \frac{6}{36} \\
&\quad + (8 - 7)^2 \cdot \frac{5}{36} + (9 - 7)^2 \cdot \frac{4}{36} + (10 - 7)^2 \cdot \frac{3}{36} + (11 - 7)^2 \cdot \frac{2}{36} + (12 - 7)^2 \cdot \frac{1}{36} \\
&= \frac{35}{6}
\end{aligned}$$

□

3. Let  $X$  be a continuous random variable with probability density

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

- (a) Find the mean of  $X$ .
- (b) Find the variance of  $X$ .
- (c) Find the third moment about the mean of  $X$ .

*Solution.* (a)

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x) dx = \int_0^1 2x^2 dx = \frac{2x^3}{3} \Big|_0^1 = \frac{2}{3}.$$

(b)

$$\sigma^2 = E(X^2) - \mu^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2 = \int_0^1 2x^3 dx - \left(\frac{2}{3}\right)^2 = \frac{x^4}{2} \Big|_0^1 - \frac{4}{9} = \frac{1}{18}$$

(c)

$$\begin{aligned}
E((X - \mu)^3) &= E(X^3) - 3\mu E(X^2) + 2\mu^3 \\
&= \int_{-\infty}^{\infty} x^3 f(x) dx - 3\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) + 2\left(\frac{2}{3}\right)^3 \\
&= \int_0^1 2x^4 dx - 1 + \frac{16}{27} \\
&= \frac{2x^5}{5} \Big|_0^1 - \frac{11}{27} \\
&= \frac{2}{5} - \frac{11}{27} \\
&= -\frac{1}{135}
\end{aligned}$$

□

4. In a certain manufacturing process, the (Fahrenheit) temperature never varies by more than  $2^\circ$  from  $62^\circ$ . The temperature is a random variable  $F$  with distribution

$x$	60	61	62	63	64
$P(F = x)$	1/10	2/10	4/10	2/10	1/10

- (a) Find the mean of and variance of  $F$ .  
(b) To convert to the measurements to degrees Celsius we let  $C = \frac{5}{9}(F - 32)$ . Find the mean and variance of  $C$ .

*Solution.* (a) Mean:

$$\mu = E(F) = \sum_x xf(x) = (60) \cdot \frac{1}{10} + (61) \cdot \frac{2}{10} + (62) \cdot \frac{4}{10} + (63) \cdot \frac{2}{10} + (64) \cdot \frac{1}{10} = 62.$$

Variance:

$$\begin{aligned}\sigma^2 &= E((F - \mu)^2) \\ &= \sum_x (x - \mu)^2 f(x) \\ &= (60 - 62)^2 \cdot \frac{1}{10} + (61 - 62)^2 \cdot \frac{2}{10} + (62 - 62)^2 \cdot \frac{4}{10} + (63 - 62)^2 \cdot \frac{2}{10} + (64 - 62)^2 \cdot \frac{1}{10} \\ &= \frac{6}{5}\end{aligned}$$

(b) Mean:

$$E(C) = E\left(\frac{5}{9}(F - 32)\right) = E\left(\frac{5}{9}F - \frac{160}{9}\right) = \frac{5}{9}E(F) - \frac{160}{9} = \frac{5}{9}\left(\frac{6}{5}\right) - \frac{160}{9} = -\frac{151}{9}.$$

Variance:

$$\text{var}(C) = \text{var}\left(\frac{5}{9}(F - 32)\right) = \text{var}\left(\frac{5}{9}F - \frac{160}{9}\right) = \left(\frac{5}{9}\right)^2 \text{var}(F) = \left(\frac{25}{81}\right)\left(\frac{6}{5}\right) = \frac{10}{27}.$$

□

5. Suppose  $X$  is a continuous random variable with probability density

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

for all  $x \in \mathbb{R}$ . It can be shown that  $X$  has moment generating function

$$M_X(t) = e^{\frac{t^2}{2}}.$$

Find the mean and variance of  $X$ .

*Solution.* Mean:

$$\mu = E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0} = \frac{d}{dt} e^{\frac{t^2}{2}} \Big|_{t=0} = t e^{\frac{t^2}{2}} \Big|_{t=0} = 0.$$

Variance:

$$E(X^2) = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = \frac{d}{dt} t e^{\frac{t^2}{2}} \Big|_{t=0} = e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}} \Big|_{t=0} = 1.$$

□