MATH1550, Winter 2023 **Names:**
Seminar 8

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1. Let X and Y be discrete random variables with the following distributions.

x	2	3	6	10
$P(X=x)$	0.2	0.2	0.5	0.1

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y	-8	-2	0	3	7
$P(Y=y)$	0.2	0.3	0.1	0.3	0.1

   (a) Find the expected value for X.
   (b) Find the expected value for Y.

Solution. (a)

$$E(X) = 2(0.2) + 3(0.2) + 6(0.5) + 10(0.1) = 5$$

(b)

$$E(Y) = (-8)(0.2) + (-2)(0.3) + 0(0.1) + 3(0.3) + 7(0.1) = -\frac{3}{5}$$

□

2. A fair coin is tossed 6 times. Let X be the number of heads that appear in the 6 tosses.
   (a) Write the probability distribution for X.
   (b) What is the expected number of heads in 6 tosses?

Solution. (a) The probability distribution for X is given by

$$f(x) = \frac{\binom{6}{x}}{64},$$

for $x \in \{0,1,2,3,4,5,6\}$ (and $f(x) = 0$ otherwise), or

x	0	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

(b)

$$E(X) = \sum_{x=0}^{6} xf(x) = 0 \cdot \frac{1}{64} + 1 \cdot \frac{6}{64} + 2 \cdot \frac{15}{64} + 3 \cdot \frac{20}{64} + 4 \cdot \frac{15}{64} + 5 \cdot \frac{6}{64} + 6 \cdot \frac{1}{64} = 3.$$

□

3. Let X and Y be a continuous random variables with probability density functions

$$f(x) = \begin{cases} \frac{1}{10}(3x^2+1) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}, \qquad p(y) = \begin{cases} \frac{1}{100} & 0 \le y \le 100 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X)$ and $E(Y)$.

Solution.

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^2 \frac{x}{10}(3x^2+1)\,dx = \frac{3x^4}{40} + \frac{x^2}{20}\Big|_0^2 = \frac{7}{5}.$$

$$E(Y) = \int_{-\infty}^{\infty} yf(y)\,dy = \int_0^{100} \frac{y}{100}\,dy = \frac{y^2}{200}\Big|_0^{100} = 50.$$

□

4. The lifetime (in years) of a certain machine component is a random variable with probability density function

$$f(x) = \begin{cases} 4(1-x)^3 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the expected lifetime of this component?

Solution.

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)\,dx \\ &= \int_0^1 4x(1-x)^3\,dx \\ &= 4\int_0^1 x - 3x^2 + 3x^3 - x^4\,dx \\ &= 4\left[\frac{x^2}{2} - x^3 + \frac{3x^4}{4} - \frac{x^5}{5}\right]_0^1 \\ &= \frac{1}{5} \end{aligned}$$

Therefore we expect such a component to last $\frac{1}{5}$ of the year. ☐

5. You arrive at a bus stop at 10:00 AM knowing that the bus will arrive some time between 10:00 AM and 10:30 AM with equal likelihood. What is your expected wait time in minutes?

Solution. Let X be the number of minutes you must wait after 10:00 AM until the next bus. The probability density function for X is

$$f(x) = \begin{cases} \frac{1}{30} & 0 \le x \le 30 \\ 0 & \text{otherwise} \end{cases}$$

The expected wait time is

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx \int_0^{30} \frac{x}{30}\,dx = \frac{x^2}{60}\Big|_0^{30} = 15,$$

i.e. we expect to wait 15 minutes. ☐

6. A pair of fair 6-sided dice are tossed. Let X be the maximum of the two numbers and Y the sum of the two numbers.

   (a) Find $E(X)$ and $E(Y)$.

   (b) Write the probability distribution for Z where $Z = X + Y$.

   (c) Verify that $E(Z) = E(X) + E(Y)$.

Solution. (a) The probability distributions for X and Y are

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

y	2	3	4	5	6	7	8	9	10	11	12
$P(Y=y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

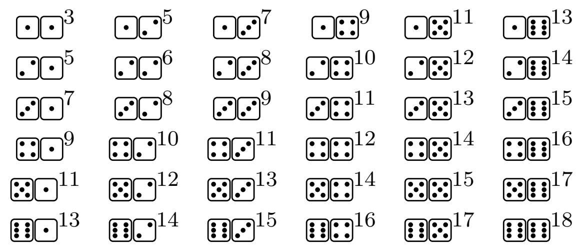
Then

$$E(X) = 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36} = \frac{161}{36},$$

and

$$E(Y) = 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + 5\cdot\frac{4}{36} + 6\cdot\frac{5}{36} + 7\cdot\frac{6}{36} + 8\cdot\frac{5}{36} + 9\cdot\frac{4}{36} + 10\cdot\frac{3}{36} + 11\cdot\frac{2}{36} + 12\cdot\frac{1}{36} = 7.$$

(b) Write the Z value beside each possible roll:



Thus the probability distribution for Z is

z	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$P(Z = z)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(c) Using the distribution from part (b) we have,

$$\begin{aligned} E(Z) &= 3 \cdot \frac{1}{36} + 5 \cdot \frac{2}{36} + 6 \cdot \frac{1}{36} + 7 \cdot \frac{2}{36} + 8 \cdot \frac{2}{36} + 9 \cdot \frac{3}{36} + 10 \cdot \frac{2}{36} + 11 \cdot \frac{4}{36} \\ &\quad + 12 \cdot \frac{3}{36} + 13 \cdot \frac{4}{36} + 14 \cdot \frac{4}{36} + 15 \cdot \frac{3}{36} + 16 \cdot \frac{2}{36} + 17 \cdot \frac{2}{36} + 18 \cdot \frac{1}{36} \\ &= \frac{413}{36} \\ &= \frac{161}{36} + 7 \\ &= E(X) + E(Y). \end{aligned}$$

□