MATH1550, Winter 2023 **Names:**
Seminar 7

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1. The joint probability distribution for discrete random variables X and Y is given in the table below.

$$\begin{array}{cc|ccc|} & & & x & \\ & & 0 & 1 & 2 \\ \hline & 0 & \frac{1}{6} & \frac{1}{4} & \frac{1}{8} \\ y & & & & \\ & 1 & \frac{1}{8} & \frac{1}{6} & \frac{1}{6} \\ \hline \end{array}$$

   (a) Verify that this is a valid joint probability distribution.

   (b) Find $P(Y = 0, X \leq 1)$.

   (c) Find the marginal distributions for X and Y.

   (d) Find the conditional probability distributions $P(X = x|Y = 0)$ and $P(Y = y|X = 1)$.

   (e) Find the conditional probabilities

      i. $P(X = 1|Y = 0)$

      ii. $P(X = 2|Y = 1)$

      iii. $P(Y = 1|X = 1)$

      iv. $P(Y = 0|X = 2)$.

   (f) Are X and Y independent?

   Solution.

   (a) We see that all probabilities are nonnegative, and that the sum of all entries in the table is 1:

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{4}{24} + \frac{6}{24} + \frac{3}{24} + \frac{3}{24} + \frac{4}{24} + \frac{4}{24} = \frac{24}{24} = 1.$$

   Therefore this is a valid joint probability distribution for X and Y.

   (b)

$$P(Y = 0, X \leq 1) = P(X = 0, Y = 0) + P(X = 1, Y = 0) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}.$$

   (c) The marginal distribution $g(x)$ for X is obtained by summing the joint distribution over all Y values, in this case the column sums. This yields

$$g(x) = \begin{cases} \frac{1}{6} + \frac{1}{8} = \frac{7}{24} & x = 0 \\ \frac{1}{4} + \frac{1}{6} = \frac{5}{12} & x = 1 \\ \frac{1}{8} + \frac{1}{6} = \frac{7}{24} & x = 2 \end{cases}$$

   Similarly the marginal distribution $h(y)$ for Y is obtained by summing the joint distribution over all X values, in this case the row sums. Thus

$$h(y) = \begin{cases} \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24} & y = 0 \\ \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24} & y = 1 \end{cases}$$

(d) Using the marginal distribution $h(y)$ for Y we have

$$P(X=x|Y=0) = \frac{P(X=x,Y=0)}{h(0)} = \begin{cases} \frac{\frac{1}{6}}{\frac{13}{24}} = \frac{4}{13} & x=0 \\ \frac{\frac{1}{4}}{\frac{13}{24}} = \frac{6}{13} & x=1 \\ \frac{\frac{1}{8}}{\frac{13}{24}} = \frac{3}{13} & x=2 \end{cases}$$

Using the marginal distribution $g(x)$ for X we have

$$P(Y=y|X=1) = \frac{P(Y=y,X=1)}{g(1)} = \begin{cases} \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5} & y=0 \\ \frac{\frac{1}{6}}{\frac{5}{12}} = \frac{2}{5} & y=1 \end{cases}$$

(e) Find the conditional probabilities

i.

$$P(X=1|Y=0) = \frac{4}{13}$$

ii.

$$P(X=2|Y=1) = \frac{P(X=2,Y=1)}{h(1)} = \frac{\frac{1}{6}}{\frac{11}{24}} = \frac{4}{11}$$

iii.

$$P(Y=1|X=1) = \frac{2}{5}$$

iv.

$$P(Y=0|X=2) = \frac{P(X=2,Y=0)}{g(2)} = \frac{\frac{1}{8}}{\frac{7}{24}} = \frac{3}{7}$$

(f) By definition X and Y are independent when $P(X=x,Y=y) = g(x)h(y)$. We see that this fails, for example

$$P(X=0,Y=0) = \frac{1}{6} \neq \frac{7}{24} \cdot \frac{13}{24} = g(0)h(0),$$

and thus X and Y are dependent.

□

2. A bag contains 40 blue marbles and 60 red marbles. Suppose 10 marbles are drawn from the bag without replacement. Let X be the number of blue marbles drawn, and Y the number of red marbles drawn.

(a) Give the joint probability distribution for X and Y.

(b) Find the marginal distributions for X and Y.

(c) Find the conditional probability distributions $P(Y=y|X=0)$ and $P(X=x|Y=3)$.

(d) Are X and Y independent?

Solution. (a)

$$f(x,y) = P(X=x,Y=y) = \begin{cases} \frac{\binom{40}{x}\binom{60}{y}}{\binom{100}{10}} & x+y=10, x,y \in \{0,\dots,10\} \\ 0 & \text{else} \end{cases}$$

(b) The marginal distribution for X is simply the probability distribution for X, which is

$$g(x) = \begin{cases} \dfrac{\binom{40}{x}\binom{60}{10-x}}{\binom{100}{10}} & x \in \{0, \dots, 10\} \\ 0 & \text{else} \end{cases}$$

Similarly the marginal distribution for Y is

$$h(y) = \begin{cases} \dfrac{\binom{60}{y}\binom{40}{10-y}}{\binom{100}{10}} & y \in \{0, \dots, 10\} \\ 0 & \text{else} \end{cases}$$

(c) Note that $X = 0$ implies $Y = 10$, otherwise $f(x, y) = 0$, so

$$P(Y = 10|X = 0) = \frac{f(0, 10)}{g(0)} = \frac{\binom{40}{0}\binom{60}{10}}{\binom{100}{10}} \cdot \frac{\binom{100}{10}}{\binom{40}{0}\binom{60}{10}} = \frac{\binom{60}{10}}{\binom{60}{10}} = 1,$$

and hence

$$P(Y = y|X = 0) = \begin{cases} 1 & y = 10 \\ 0 & \text{else} \end{cases}$$

Similarly $Y = 3$ implies $X = 7$ so

$$P(X = x|Y = 3) = \begin{cases} 1 & x = 7 \\ 0 & \text{else} \end{cases}$$

(d) Note that

$$f(10, 0) = \frac{\binom{40}{10}\binom{60}{0}}{\binom{100}{10}} \neq \frac{\binom{40}{10}\binom{60}{0}}{\binom{100}{10}} \frac{\binom{60}{0}\binom{40}{10}}{\binom{100}{10}} = g(10)h(0)$$

which demonstrates that X and Y are not independent. Clearly the probability of X depends on the value for Y and vice versa.

□

3. Let X and Y be joint continuous random variables with joint probability density function given below.

$$f(x, y) = \begin{cases} x + Cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

(a) Determine an appropriate value for $C \in \mathbb{R}$ (if one exists).

(b) Find $P(0 \leq X \leq \frac{1}{2}, \frac{1}{2} \leq Y \leq 1)$.

(c) Find the joint cumulative distribution function $F(x, y)$ for X and Y.

(d) Find the marginal distributions for X and Y.

(e) Find the conditional probability distributions $P(Y = y|X = 0)$ and $P(X = x|Y = 0.5)$.

(f) Are X and Y independent?

Solution. (a) Solve

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\;dx\;dy \\
&= \int_0^1\int_0^1 x + Cy^2\;dx\;dy \\
&= \int_0^1 \frac{x^2}{2} + Cxy^2\Big|_0^1\;dy \\
&= \int_0^1 \frac{1}{2} + Cy^2\;dy \\
&= \frac{y}{2} + C\frac{y^3}{3}\Big|_0^1 \\
&= \frac{1}{2} + \frac{C}{3}
\end{aligned}
$$

Thus

$$1 = \frac{1}{2} + \frac{C}{3} \quad\Rightarrow\quad \frac{C}{3} = \frac{1}{2} \quad\Rightarrow\quad C = \frac{3}{2}.$$

Note also that $f(x,y) \geq 0$ for this value for C.

(b)

$$
\begin{aligned}
P\left(0 \leq X \leq \frac{1}{2}, \frac{1}{2} \leq Y \leq 1\right) &= \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 x + \frac{3y^2}{2}\;dy\;dx \\
&= \int_0^{\frac{1}{2}} xy + \frac{y^3}{2}\Big|_{\frac{1}{2}}^1\;dx \\
&= \int_0^{\frac{1}{2}} \frac{x}{2} + \frac{7}{16}\;dx \\
&= \frac{x^2}{4} + \frac{7x}{16}\Big|_0^{\frac{1}{2}} \\
&= \frac{1}{16} + \frac{7}{32} \\
&= \frac{9}{32} \\
&= 0.28125
\end{aligned}
$$

(c) For $x < 0$ or $y < 0$ we have $F(x,y) = 0$.

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(s,t)\;dt\;ds = \int_{-\infty}^{x}\int_{-\infty}^{y} 0\;dt\;ds = 0.$$

For $0 \le x \le 1$, $0 \le y \le 1$ we have

$$\begin{aligned} F(x,y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\,dt\,ds \\ &= \int_{0}^{x} \int_{0}^{y} s + \frac{3t^2}{2}\,dt\,ds \\ &= \int_{0}^{x} st + \frac{t^3}{2}\Big|_0^y\,ds \\ &= \int_{0}^{x} sy + \frac{y^3}{2}\,ds \\ &= \frac{s^2y}{2} + \frac{sy^3}{2}\Big|_0^x \\ &= \frac{x^2y}{2} + \frac{xy^3}{2}. \end{aligned}$$

For $x > 1$, $0 \le y \le 1$ we have

$$\begin{aligned} F(x,y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\,dt\,ds \\ &= \int_{0}^{1} \int_{0}^{y} s + \frac{3t^2}{2}\,dt\,ds \\ &= \int_{0}^{1} st + \frac{t^3}{2}\Big|_0^y\,ds \\ &= \int_{0}^{1} sy + \frac{y^3}{2}\,ds \\ &= \frac{s^2y}{2} + \frac{sy^3}{2}\Big|_0^1 \\ &= \frac{y}{2} + \frac{y^3}{2}. \end{aligned}$$

For $y > 1$, $0 \le x \le 1$ we have

$$\begin{aligned} F(x,y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\,dt\,ds \\ &= \int_{0}^{x} \int_{0}^{1} s + \frac{3t^2}{2}\,dt\,ds \\ &= \int_{0}^{x} st + \frac{t^3}{2}\Big|_0^1\,ds \\ &= \int_{0}^{x} s + \frac{1}{2}\,ds \\ &= \frac{s^2}{2} + \frac{s}{2}\Big|_0^x \\ &= \frac{x^2}{2} + \frac{x}{2}. \end{aligned}$$

For $x > 1$, $y > 1$ we have

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\,dt\,ds = \int_{0}^{1} \int_{0}^{1} s + \frac{3t^2}{2}\,dt\,ds = 1.$$

In summary

$$F(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ \frac{x^2y}{2} + \frac{xy^3}{2} & 0 \le x \le 1, 0 \le y \le 1 \\ \frac{y}{2} + \frac{y^3}{2} & x > 1, 0 \le y \le 1 \\ \frac{x^2}{2} + \frac{x}{2} & y > 1, 0 \le x \le 1 \\ 1 & x > 1, y > 1 \end{cases}$$

(d) Marginal distribution for X: For $0 \le x \le 1$,

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x,y)\, dy \\ &= \int_0^1 x + \frac{3y^2}{2}\, dy \\ &= xy + \frac{y^3}{2}\Big|_0^1 \\ &= x + \frac{1}{2} \end{aligned}$$

and $g(x) = 0$ otherwise. Marginal distribution for Y: For $0 \le y \le 1$,

$$\begin{aligned} h(y) &= \int_{-\infty}^{\infty} f(x,y)\, dx \\ &= \int_0^1 x + \frac{3y^2}{2}\, dx \\ &= \frac{x^2}{2} + \frac{3xy^2}{2}\Big|_0^1 \\ &= \frac{1}{2} + \frac{3y^2}{2} \end{aligned}$$

and $h(y) = 0$ otherwise.

(e) For $0 \le y \le 1$,

$$P(Y = y|X = 0) = \frac{f(0,y)}{g(0)} = \frac{\frac{3y^2}{2}}{\frac{1}{2}} = 3y^2,$$

thus

$$P(Y = y|X = 0) = \begin{cases} 3y^2 & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}.$$

For $0 \le x \le 1$,

$$P(X = x|Y = 0.5) = \frac{f(x, 0.5)}{h(0.5)} = \frac{x + \frac{3}{8}}{\frac{7}{8}} = \frac{8x+3}{7},$$

thus

$$P(X = x|Y = 0.5) = \begin{cases} \frac{8x+3}{7} & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}.$$

(f) Note that for $0 \le x \le 1$ and $0 \le y \le 1$ we have

$$f(x,y) = x + \frac{3y^2}{2}$$

whereas

$$g(x)h(y) = \left(x + \frac{1}{2}\right)\left(\frac{1}{2} + \frac{3y^2}{2}\right) = \frac{x}{2} + \frac{3xy^2}{2} + \frac{1}{4} + \frac{3y^2}{4}.$$

We can see that $f(x,y) \neq g(x)h(y)$ for all $0 \leq x \leq 1$ and $0 \leq y \leq 1$, for example

$$f(0,1) = \frac{3}{2} \neq 1 = g(0)h(1).$$

Therefore X and Y are dependent.

$\square$