**MATH1550, Winter 2023** **Names:**
**Seminar 6**

---

1. Find the cumulative distribution function $F(x)$ for each probability density function $f(x)$.

(a)
$$f(x) = \begin{cases} \frac{3}{8}(4x - 2x^2) & \text{for } x \in [0,2] \\ 0 & \text{else} \end{cases}$$

(b)
$$f(x) = \begin{cases} \frac{10}{x^2} & \text{for } x > 10 \\ 0 & \text{else} \end{cases}$$

(c)
$$f(x) = \begin{cases} 10(x^3 - x^4) & \text{for } x \in [0,1] \\ \frac{4}{3x^3} & \text{for } x \in (1,2] \\ 0 & \text{else} \end{cases}$$

*Solution.* (a) For $x < 0$
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0.$$

For $0 \leq x \leq 2$
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} \frac{3}{8}(4t - 2t^2)\,dt = \frac{3}{8}\left(2t^2 - \frac{2t^3}{3}\right)\Big|_0^x = \frac{3x^2}{4} - \frac{x^3}{4}.$$

For $x \geq 2$
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{2} \frac{3}{8}(4t - 2t^2)\,dt + \int_{2}^{x} 0\,dt = \frac{3}{8}\left(2t^2 - \frac{2t^3}{3}\right)\Big|_0^2 = 1.$$

Thus
$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{3x^2}{4} - \frac{x^3}{4} & \text{for } 0 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

(b) For $x \leq 10$
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0.$$

For $x > 10$
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{10} 0\,dt + \int_{10}^{x} \frac{10}{t^2}\,dt = -\frac{10}{t}\Big|_{10}^{x} = 1 - \frac{10}{x}.$$

Thus
$$F(x) = \begin{cases} 0 & \text{for } x \leq 10 \\ 1 - \frac{10}{x} & \text{for } x > 10 \end{cases}$$

(c) For $x < 0$
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0.$$

For $0 \leq x \leq 1$
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} 10(t^3 - t^4)\,dt = \frac{5t^4}{2} - 2t^5\Big|_0^x = \frac{5x^4}{2} - 2x^5.$$

For $1 < x \leq 2$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{1} f(t)\,dt + \int_{1}^{x} \frac{4}{3t^3}\,dt = F(1) + \left[-\frac{2}{3t^2}\right]_1^x = \frac{1}{2} + \left(\frac{2}{3} - \frac{2}{3x^2}\right) = \frac{7}{6} - \frac{2}{3x^2}.$$

For $x > 2$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{2} f(t)\,dt + \int_{2}^{x} 0\,dt = F(2) = 1.$$

Thus

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{5x^4}{2} - 2x^5 & \text{for } 0 \leq x < 1 \\ \frac{7}{6} - \frac{2}{3x^2} & \text{for } 1 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

□

2. The cumulative distribution function $F(x)$ for a continuous random variable $X$ is given. Find the probability density $f(x)$ for $X$.

(a)

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

(b)

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^2}{2} & \text{for } 0 \leq x \leq 1 \\ 2x - \frac{x^2}{2} - 1 & \text{for } 1 < x \leq 2 \\ 1 & \text{for } x > 2 \end{cases}$$

(c)

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases}$$

*Solution.* To find $f(x)$ take $\frac{d}{dx}F(x)$.

(a)

$$f(x) = \begin{cases} 2x & \text{for } 0 \leq x < 1 \\ 0 & \text{else} \end{cases}$$

(b)

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 2 - x & \text{for } 1 < x \leq 2 \\ 0 & \text{else} \end{cases}$$

(c)

$$f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases}$$

□

3. A fair coin is tossed 4 times. You win \$3 if 2 or 4 heads appear, you win \$1 if 1 or 3 heads appear and you lose \$6 if if no heads appear. Let $X$ be the number of heads, and $Y$ the number of dollars won, after 4 tosses. Give the joint probability distribution $f(x, y)$, for $X$ and $Y$.

*Solution.*

| $y$ \ $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| -6 | $\frac{1}{16}$ | | | | |
| 1 | | $\frac{4}{16}$ | | $\frac{4}{16}$ | |
| 3 | | | $\frac{6}{16}$ | | $\frac{1}{16}$ |

□

4. Two fair 6-sided dice are thrown. Let $X$ be the largest value appearing on either die, and $Y$ be value appearing on the first die. Give the joint probability distribution $f(x, y)$, for $X$ and $Y$.

*Solution.*

| $y$ \ $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ |
| 2 | | $\frac{2}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ |
| 3 | | | $\frac{3}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ |
| 4 | | | | $\frac{4}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ |
| 5 | | | | | $\frac{5}{36}$ | $\frac{1}{36}$ |
| 6 | | | | | | $\frac{6}{36}$ |

□

5. A fair coin is tossed three times. Let $X$ be the number of heads that appear, and $Y$ the toss (1, 2 or 3) where heads first appears, or $Y = 0$ if heads dose not appear. Give the joint probability distribution $f(x, y)$, for $X$ and $Y$.

*Solution.*

| $y$ \ $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\frac{1}{8}$ | | | |
| 1 | | $\frac{1}{8}$ | $\frac{2}{8}$ | $\frac{1}{8}$ |
| 2 | | $\frac{1}{8}$ | $\frac{1}{8}$ | |
| 3 | | $\frac{1}{8}$ | | |

□

6. The joint probability distribution for discrete random variables $X$ and $Y$ is given in the table below.

| $y$ \ $x$ | 1 | 2 | 3 |
|---|---|---|---|
| 0 | $\frac{1}{8}$ | $k$ | $\frac{1}{8}$ |
| 2 | $\frac{1}{8}$ | $\frac{1}{24}$ | $\frac{1}{6}$ |
| 4 | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{12}$ |

(a) Determine an appropriate value for $k \in \mathbb{R}$.

(b) Find $P(X = 1, Y = 4)$.

(c) Find $P(X \leq 2.25, Y \leq 3)$.

(d) Find $P(X \leq 2.6, Y > 1)$.

*Solution.* (a) In order that all probabilities sum to 1, we must have $k = \frac{1}{12}$

(b)

$$P(X = 1, Y = 4) = \frac{1}{6}.$$

(c)

$$P(X \leq 2.25, Y \leq 3) = P(1,0) + P(2,0) + P(1,2) + P(2,2) = \frac{1}{8} + \frac{1}{12} + \frac{1}{8} + \frac{1}{24} = \frac{3}{8}.$$

(d)

$$P(X \leq 2.6, Y > 1) = P(1,2) + P(2,2) + P(1,4) + P(2,4) = \frac{1}{8} + \frac{1}{24} + \frac{1}{6} + \frac{1}{12} = \frac{5}{12}.$$

□