MATH1550, Winter 2023 **Names:**
Seminar 5

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1. A continuous random variable X is said to have *uniform density* (or *equiprobable density*) if its probability density function is constant on some region and zero elsewhere; for example if $C \in \mathbb{R}$, then

$$f(x) = \begin{cases} C & \text{for } x \in [a, b] \\ 0 & \text{else} \end{cases}$$

is such a function. The set $[a, b]$, where f is nonzero, is called the *support* of f. This type of probability density function models the situation of having an equally likely chance of choosing a number at random from the interval $[a, b]$.

(a) Let X be a number chosen at random from the interval $[3, 15]$. Give the probability density function for X assuming it has uniform density.

(b) Use the probability density function from part (a) to compute:

i. $P(7 \leq X \leq 10)$
ii. $(X < 10)$
iii. $P(0 \leq X \leq 20)$
iv. $P(1 \leq X < 4)$
v. $P(X^2 - 2X - 9 < 0)$

(c) What must the value for C be for the uniform density function f with support $[a, b]$?

(d) In this case, give a formula for finding $P(c \leq X \leq d)$ where $a \leq c \leq d \leq b$.

Solution. (a) We have

$$f(x) = \begin{cases} C & \text{for } x \in [3, 15] \\ 0 & \text{else} \end{cases}$$

We require that

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_3^{15} C\, dx = Cx\Big|_3^{15} = 15C - 3C = 12C,$$

which implies $C = \frac{1}{12}$. Thus

$$f(x) = \begin{cases} \frac{1}{12} & \text{for } x \in [3, 15] \\ 0 & \text{else} \end{cases}$$

(b) i.

$$P(7 \leq X \leq 10) = \int_7^{10} \frac{1}{12}\, dx = \frac{x}{12}\Big|_7^{10} = \frac{10 - 7}{12} = \frac{5}{12}.$$

ii.

$$P(X < 10) = \int_{-\infty}^{10} f(x)\, dx = \int_{-\infty}^{3} 0\, dx + \int_3^{10} \frac{1}{12}\, dx = 0 + \frac{x}{12}\Big|_3^{10} = \frac{7}{12}.$$

iii.

$$P(0 \leq X \leq 20) = \int_0^3 0\, dx + \int_3^{15} \frac{1}{12}\, dx + \int_{15}^{20} 0\, dx = 0 + \frac{x}{12}\Big|_3^{15} + 0 = \frac{12}{12} = 1.$$

iv.

$$P(1 \leq X < 4) = \int_1^3 0\, dx + \int_3^4 \frac{1}{12}\, dx = 0 + \frac{x}{12}\Big|_3^4 = \frac{1}{12}.$$

v.

$$\begin{aligned} P(X^2 - 2X - 8 < 0) &= P((X-1)^2 - 9 < 0) \quad \text{(complete the square)} \\ &= P((X-1)^2 < 9) \\ &= P(-3 < X - 1 < 3) \\ &= P(-2 < X < 4) \\ &= \int_3^4 \frac{1}{12}\, dx \\ &= \frac{1}{12}. \end{aligned}$$

(c) Since

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_a^b C\, dx = Cx\Big|_a^b = C(b-a),$$

we see that

$$C = \frac{1}{b-a}$$

(assuming $a < b$).

(d) Based on the previous examples it is easy to see that

$$P(c \le X \le d) = \frac{d-c}{b-a}$$

of course we can derive this as

$$P(c \le X \le d) = \int_c^d \frac{1}{b-a}\, dx = \frac{x}{b-a}\Big|_c^d = \frac{d-c}{b-a}.$$

□

2. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{C}{x^3} & \text{for } x \in [a,b] \\ 0 & \text{else} \end{cases}$$

where $C \in \mathbb{R}$ and $a < b$.

(a) Find an appropriate value for C if $a = 1$ and $b = 2$.

(b) Find an appropriate value for C for arbitrary a and b with $0 < a < b$.

Solution. (a) We require that

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 \frac{C}{x^3}\, dx = -\frac{C}{2x^2}\Big|_0^1 = -\frac{C}{8} + \frac{C}{2} = \frac{-C + 4C}{8} = \frac{3C}{8},$$

which implies

$$C = \frac{8}{3}.$$

(b) We require that

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_a^b \frac{C}{x^3}\, dx = -\frac{C}{2x^2}\Big|_a^b = \frac{C}{2a^2} - \frac{C}{2b^2} = \frac{C(b^2 - a^2)}{2a^2 b^2},$$

which implies

$$C = \frac{2a^2b^2}{b^2 - a^2}.$$

□

3. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} C(4x - 2x^2) & \text{for } x \in [0,2] \\ 0 & \text{else} \end{cases}$$

   (a) Show that $C = \frac{3}{8}$.

   (b) Compute $P(X > 1)$.

Solution. (a) We have that

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 C(4x - 2x^2)\,dx = C\left[2x^2 - \frac{2x^3}{3}\right]_0^2 = \frac{8}{3}C,$$

which implies $C = \frac{3}{8}$.

(b)

$$P(X > 1) = \int_1^{\infty} f(x)\,dx = \int_1^2 \frac{3}{8}(4x - 2x^2)\,dx = \frac{3}{8}\left[2x^2 - \frac{2x^3}{3}\right]_1^2 = \frac{3}{8}\left[\frac{8}{3} - \frac{4}{3}\right] = \frac{1}{2}.$$

□

4. The lifetime (in years) of a certain machine component is a random variable with probability density function

$$f(x) = \begin{cases} \frac{10}{x^2} & \text{for } x > 10 \\ 0 & \text{else} \end{cases}$$

   (a) What is the probability that such a component will last more than 20 years?.

   (b) A machine contains 6 such components. The lifetime of any component is unaffected by the others. What is the probability that exactly 4 of them last over 15 years? Hint: Let A_i be the event that component i will last more then 15 years, and assume that events $A_1, A_2, A_3, A_4, A_5, A_6$ are independent.

Solution. (a)

$$P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2}\,dx = \lim_{n\to\infty} -\frac{10}{x}\bigg|_{20}^{n} = \lim_{n\to\infty} -\frac{10}{n} - \left(-\frac{10}{20}\right) = \frac{1}{2}.$$

(b) The probability that component i will last more that 15 years is

$$P(A_i) = \int_{15}^{\infty} \frac{10}{x^2}\,dx = \lim_{n\to\infty} -\frac{10}{x}\bigg|_{15}^{n} = \lim_{n\to\infty} -\frac{10}{n} - \left(-\frac{10}{15}\right) = \frac{2}{3}$$

which means that the probability it fails before 15 years is

$$P(A_i') = 1 - P(A_i) = \frac{1}{3}.$$

As an example, the probability that components 1, 2, 3 and 4 last longer than 15 years, while components 5 and 6 fail before 15 years is

$$
\begin{aligned}
P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5' \cap A_6') &= P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) \cdot P(A_5') \cdot P(A_6') \\
&= \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 \\
&= \frac{16}{729},
\end{aligned}
$$

using that fact that these events are independent. It follows that the probability that any 4 of 6 last more that 15 years, while the other 2 fail, is then

$$
\binom{6}{4} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 = \frac{80}{243}.
$$

□

5. You arrive at a bus stop at 10:00 AM knowing that the bus will arrive some time between 10:00 AM and 10:30 AM with equal likelihood.
   (a) What is the probability that you will have to wait longer than 10 minutes for the bus to arrive?
   (b) If at 10:15 AM the bus has not arrived, what is the probability that you will have to wait at least 10 more minutes?

Solution. (a) let X be the number of minutes after 10:00 AM until the bus arrives. The probability density function for X is

$$
f(x) = \begin{cases} \frac{1}{30} & \text{for } x \in [0, 30] \\ 0 & \text{else} \end{cases}
$$

Then

$$
P(X > 10) = \int_{10}^{30} \frac{1}{30}\, dx = \left.\frac{x}{30}\right|_{10}^{30} = \frac{2}{3},
$$

or, using what we know about uniform probability densities,

$$
P(X > 10) = P(10 \le X \le 30) = \frac{30 - 10}{30} = \frac{2}{3}.
$$

(b) Let A be event that $X \ge 15$ and B the event that $X \ge 25$, then we are interested in

$$
P(B|A) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(A)} = \frac{\frac{30-25}{30}}{\frac{30-15}{30}} = \frac{1}{3}.
$$

□

6. Determine whether or not the following function may serve as a valid probability density function.

$$
f(x) = \begin{cases} \frac{4}{3}(2x - x^3) & \text{for } \frac{1}{2} \le x \le \frac{3}{2} \\ 0 & \text{else} \end{cases}
$$

Solution. Note that

$$
f\left(\frac{3}{2}\right) = -\frac{1}{2} < 0
$$

therefore $f(x)$ cannot serve as a valid probability density function. We see see however that

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{4}{3}(2x - x^3)\,dx = \frac{4}{3}\left[x^2 - \frac{x^4}{4}\right]_{\frac{1}{2}}^{\frac{3}{2}} = \frac{4}{3}\left[\frac{63}{64} - \frac{15}{64}\right] = 1.$$

□