MATH1550, Winter 2023 **Names:**
Seminar 4

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1. You are given two identical jars and 50 balls, which are identical except for colour with 25 green and 25 yellow. The balls will be distributed into the jars, after which a jar will be chosen at random and then a ball drawn at random from that jar. How should one distribute the balls into the jars to have the greatest possible chance that the ball drawn from the chosen jar is yellow?

   Solution. Suppose jar 1 has y yellow balls, and hence jar 2 has $25 - y$ yellow balls. By the rule of total probability, the probability of drawing a yellow ball is

$$\begin{aligned} P(\text{yellow ball}) &= P(\text{Jar } 1)P(\text{yellow ball}|\text{Jar } 1) + P(\text{Jar } 2)P(\text{yellow ball}|\text{Jar } 2) \\ &= \left(\frac{1}{2}\right)\left(\frac{y}{25}\right) + \left(\frac{1}{2}\right)\left(\frac{25-y}{25}\right) \\ &= \frac{y + 25 - y}{50} \\ &= \frac{25}{50}. \end{aligned}$$

   Therefore, it doesn't matter how the yellow balls are distributed, the probability will always be $\frac{1}{2}$. □

2. A certain university class is made up of 11 freshman, 19 sophomores, 14 juniors and 6 seniors. One student is selected at random.

   (a) What is the sample space of this experiment?

   (b) Let X represent chosen student's year of study. Write the range of X.

   (c) Write the probability distribution for X.

   Solution.

   (a) We may take the sample space to be the set of 50 students in this class, or if we are only interested in the year of study, we might take the sample space to be the set $\{\text{freshman}, \text{sophomore}, \text{junior}, \text{senior}\}$.

   (b) The range of X is $\{1, 2, 3, 4\}$.

   (c) The probability distribution for X is

$$P(X = x) = \begin{cases} \frac{11}{50} & \text{for } x = 1 \\ \frac{19}{50} & \text{for } x = 2 \\ \frac{14}{50} & \text{for } x = 3 \\ \frac{6}{50} & \text{for } x = 4 \end{cases}$$

   □

3. The UTF-8 binary coding system encodes written characters with an 8-bit number. The conversion for Upper case letters in the English alphabet is shown in the table below.

A: 010000001	B: 010000010	C: 01000011	D: 01000100	E: 01000101	F: 01000110
G: 01000111	H: 01001000	I: 01001001	J: 01001010	K: 01001011	L: 01001100
M: 01001101	N: 01001110	O: 01001111	P: 01010000	Q: 01010001	R: 01010010
S: 01010010	T: 01010010	U: 01010101	V: 01010110	W: 01010111	X: 01011000
Y: 01011001	Z: 01011010				

A letter is selected at random from a book. Let X be the random variable which gives the number of 1's in an 8-bit code for a letter from A to Z.

(a) Write the range of X.

(b) The relative frequency of use in texts for each letter in the English alphabet is given in the following table.

A: 8.1%	B: 1.5%	C: 2.78%	D: 4.3%	E: 12.9%	F:2.2%
G: 2%	H: 6.1%	I: 7%	J: 0.15%	K: 0.77%	L: 4%
M: 2.4%	N: 6.7%	O: 7.4%	P: 1.9%	Q: 0.095%	R: 6%
S: 6.25%	T: 9.05%	U: 2.8%	V: 0.98%	W: 2.4%	X: 0.15%
Y: 2%	Z: 0.075%				

Give the probability distribution for X.

Solution.

(a) The range of X is $\{2, 3, 4, 5\}$.

(b) The probability distribution for X is

$$P(X = 2) = P(\{\text{A,B,D,H,P}\}) = 0.219$$

$$P(X = 3) = P(\{\text{C,E,F,I,J,L,Q,R,S,T,X}\}) = 0.50575$$

$$P(X = 4) = P(\{\text{G,K,M,N,U,V,Y,Z}\}) = 0.17725$$

$$P(X = 5) = P(\{\text{O,W})\} = 0.098$$

□

4. A box contains 5 colored balls, 2 black and 3 white. Balls are drawn successively without replacement. If the random variable X is the number of draws until the last black ball is obtained, find the probability distribution of X.

Solution. The sample space of for this experiment is

$$S = \{BB, BWB, WBB, BWWB, WBWB, WWBB, BWWWB, WBWWB, WWBWB, WWWBB\},$$

and hence the range of X is $\{2, 3, 4, 5\}$. The probability distribution for X is

$$P(X = 2) = P(\{BB\}) = \left(\frac{2}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{10}$$

$$P(X = 3) = P(\{\text{BWB}\}) + P(\{\text{WBB}\}) = \left(\frac{2}{5}\right)\left(\frac{3}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{3}\right) = \frac{1}{5}$$

$$\begin{aligned} P(X = 4) &= P(\{\text{BWWB}\}) + P(\{\text{WBWB}\}) + P(\{\text{WWBB}\}) \\ &= \left(\frac{2}{5}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) \\ &= \frac{3}{10} \end{aligned}$$

$$
\begin{aligned}
P(X=5) &= P(\{\text{BWWWB}\}) + P(\{\text{WBWWB}\}) + P(\{\text{WWBWB}\}) + P(\{\text{WWWBB}\}) \\
&= \left(\frac{2}{5}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{1}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{1}\right) \\
&+ \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{1}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{3}\right)\left(\frac{2}{2}\right)\left(\frac{1}{1}\right) \\
&= \frac{2}{5}
\end{aligned}
$$

Alternative: A simpler way to arrive at the same probability distribution, is to notice that all 10 outcomes in S are equally likely. To see this, imagine that instead all 5 balls are always drawn but that we keep track of the order of the colours. Then

$$
\begin{aligned}
S = \{&BBWWW, BWBWW, WBBWW, BWWBW, WBWBW, \\
&WWBBW, BWWWB, WBWWB, WWBWB, WWWBB\}.
\end{aligned}
$$

Since each ball has an equally likely change of being drawn at any time, it follows that these 10 outcomes are equally likely, and the probability distribution above follows. $\square$

5. Let X be a random variable whose range is the whole numbers from 1 to 12. Show that the following is a valid probability distribution for X, then give the cumulative distribution function for X.

$$P(X = x) = \frac{2x-1}{144}$$

Solution. It is easy to see that $P(X = x) \geq 0$ for each $x \in \{1, \ldots, 12\}$. It suffices then to show that $P(\{1, \ldots, 12\}) = 1$. We can simply plug in each value for x add up the individual probabilities, or we can compute using properties of sums as follows,

$$
\begin{aligned}
P(\{1, \ldots, 12\}) &= \sum_{x=1}^{12} P(X = x) \\
&= \sum_{x=1}^{12} \frac{2x-1}{144} \\
&= \frac{1}{144} \sum_{x=1}^{12} (2x-1) \\
&= \frac{2}{144} \sum_{x=1}^{12} x - \frac{1}{144} \sum_{x=1}^{12} 1 \\
&= \frac{2}{144} \left(\frac{12(13)}{2}\right) - \frac{1}{144}(12) \\
&= \frac{12 \cdot (13) - 12}{144} \\
&= \frac{12 \cdot (12)}{144} \\
&= 1.
\end{aligned}
$$

Note that we have made use the following summation property with $n = 12$:

$$\sum_{x=1}^{n} x = \frac{n(n+1)}{2}.$$

□

6. The cumulative distribution function for a discrete random variable X is given by

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ 0.25 & \text{for } -1 \leq x < 1 \\ 0.35 & \text{for } 1 \leq x < 3 \\ 0.71 & \text{for } 3 \leq x < 5 \\ 1 & \text{for } x \geq 5 \end{cases}$$

(a) Write the range of X.

(b) Find $P(X \leq 3)$ and $P(X = 3)$.

(c) Give the probability distribution for X.

(d) Find $P(X < 3)$, and $P(X \geq 1)$.

Solution.

(a) The range of X is $\{-1, 1, 3, 5\}$.

(b) Using the cumulative distribution we have

$$P(X \leq 3) = F(3) = 0.71,$$

and

$$P(X = 3) = F(3) - F(1) = 0.71 - 0.35 = 0.36.$$

(c) The probability distribution for X is

$$P(X = x) = \begin{cases} 0.25 & \text{for } x = -1 \\ 0.10 & \text{for } x = 1 \\ 0.36 & \text{for } x = 3 \\ 0.29 & \text{for } x = 5 \end{cases}$$

(d) Using the probability distribution we have

$$P(X < 3) = P(X = -1) + P(X = 1) = 0.25 + 0.10 = 0.35,$$

and

$$P(X \geq 1) = P(X = 1) + P(X = 3) + P(X = 5) = 0.10 + 0.36 + 0.29 = .75.$$

Alternatively, using the cumulative distribution we have

$$P(X < 3) = F(1) = 0.35,$$

and

$$P(X \geq 1) = 1 - P(X < 1) = 1 - F(-1) = 1 + 0.25 = 0.75.$$

□