MATH1550, Winter 2023 **Names:**
Seminar 1

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1. How many different words can be made using the letters in each of the following. For this question you must use all of the letters, but the "words" you make don't need to be in the dictionary

   (a) MATH
   (b) ALGEBRA
   (c) ANALYSIS
   (d) PROBABILITY
   (e) COMBINATORICS
   (f) GEOMETRY
   (g) STATISTICS
   (h) TOPOLOGY
   (i) DIFFERENTIAL EQUATIONS

Solution. Let n be the numbers of letters (including repetitions) in the given word, and n_{α_i} be the number of times the letter "α_i" appears. Then $n = n_{\alpha_1} + n_{\alpha_2} + \cdots + n_{\alpha_k}$, assuming there are k different letters present. The number of different words that can be made is

$$\frac{n!}{n_{\alpha_1}! \cdot n_{\alpha_2}! \cdot \ \ldots \ \cdot n_{\alpha_k}!}.$$

For example in part (a), $n = 4$, and $n_M = n_A = n_T = n_H = 1$. Thus there are

$$\frac{4!}{1! \cdot 1! \cdot 1! \cdot 1!} = \frac{24}{1} = 24$$

different words to be made. Since $1! = 1$, we won't bother to write those in the denominator next time. In part (b) $n = 7$ and $n_A = 2$, $n_L = n_G = n_E = n_B = n_R = 1$. Thus there are

$$\frac{7!}{2!} = \frac{5040}{2} = 2520$$

different words to be made. The remaining parts are solved in the same way.

Answers:

(a)
$$4! = 24$$

(b)
$$\frac{7!}{2!} = \frac{5040}{2} = 2520$$

(c)
$$\frac{8!}{2! \cdot 2!} = \frac{40320}{4} = 10080$$

(d)
$$\frac{11!}{2! \cdot 2!} = \frac{399168800}{4} = 9979200$$

(e)
$$\frac{13!}{2! \cdot 2! \cdot 2!} = \frac{6227020800}{8} = 778377600$$

(f)
$$\frac{8!}{2!} = \frac{40320}{2} = 20160$$

(g)
$$\frac{10!}{3! \cdot 3! \cdot 2!} = \frac{3628800}{72} = 50400$$

(h)
$$\frac{8!}{3!} = \frac{40320}{72} = 6720$$

(i) Not including a space

$$\frac{21!}{3!2!3!2!2!2!} = \frac{51090942171709440000}{576} = 88699522381440000$$

Including a space (arrange all letters then place the space between any two letters)

$$\frac{21!}{3!2!3!2!2!2!} \cdot 20 = 88699522381440000 \cdot 20 = 1773991047628800000$$

□

2. A multiple choice test consists of 12 questions. How many different ways can a student complete the test if:
   (a) There are 3 possible answers to each questions?
   (b) Half of the problems have 3 possible answers and the other half have 4 possible answers?

Solution. (a) There are
$$3^{12} = 531441$$
different ways that the test could be completed (refer to general counting rule for compound events).

(b) There are
$$3^6 \cdot 4^6 = 2985984$$
different ways that the test could be completed in this case.

□

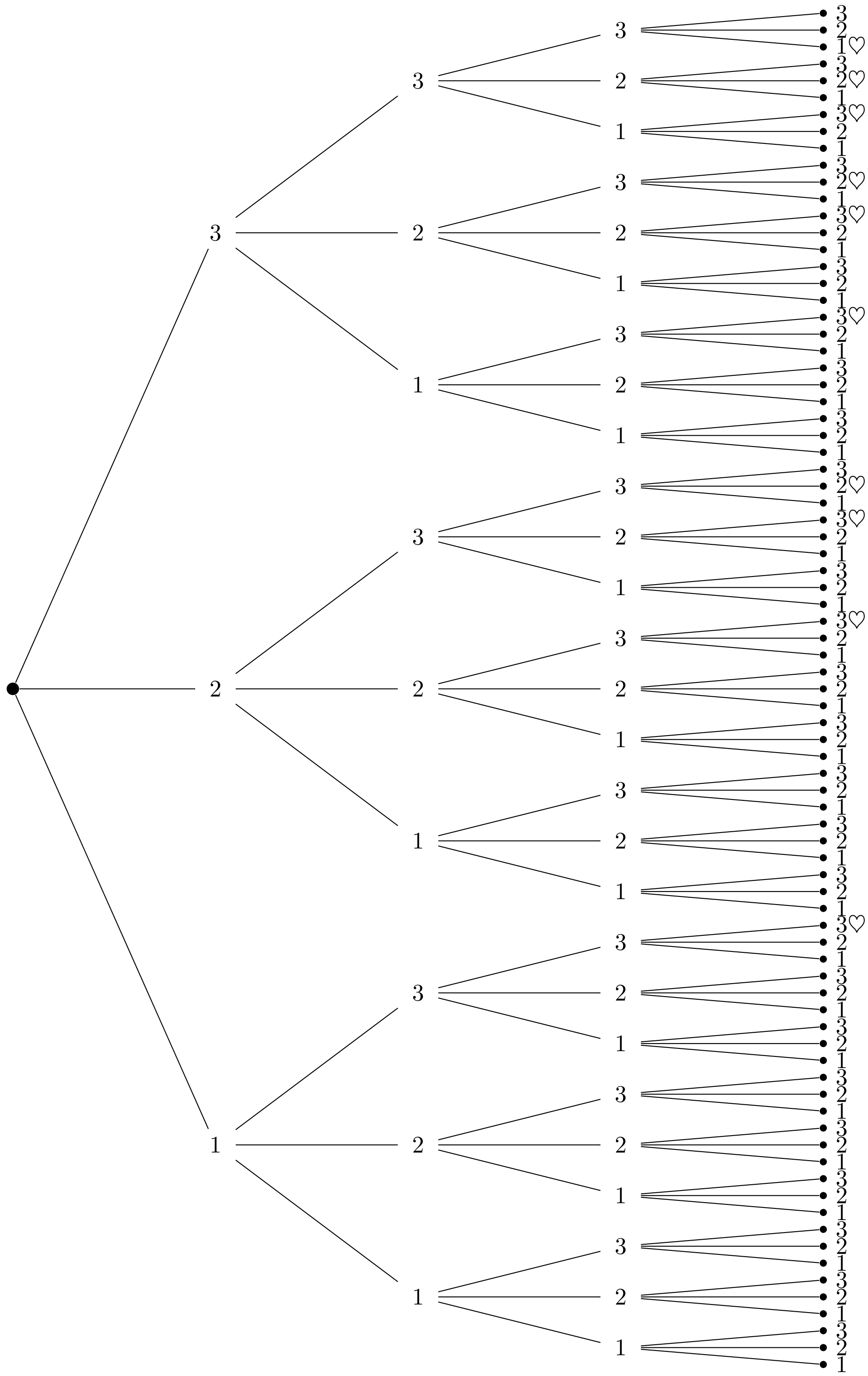
3. At "The Smoothie Shack" you can choose to have either 1, 2, 3 or 4 different fruits to blend into your smoothie. The fruits you can choose from are: banana, kiwi, mango and pineapple. How many different flavour combinations are possible?

Solution. Recall that
$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$
(also denoted ${}_nC_i$) is the number of ways that one can choose i things from a selection of n things without considering the order in which the i things are chosen; i.e. this is the number of subsets of size i made from a set of size n.

We add the number of numbers of flavour combinations that can be made by choosing 1, 2, 3, or 4 fruits respectively. This gives us
$$\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 4 + 6 + 4 + 1 = 15$$
different smoothie flavours. □

4. Every day a certain mathematics professor drinks either 1, 2 or 3 shots of espresso. Draw a tree diagram that you can use to count the number of ways this person can drink exactly 10 shots of espresso from Monday to Thursday *(Friday he drinks tea)*.

Solution.



□

5. A flag with 4 vertical bars is to be constructed as shown below. There are 8 different colors to choose for each bar.

   (a) How many flags can be made?
   (b) How many flags can be made if no two bars can have the same color?
   (c) How many flags can be made if only the middle two bars may be the same color?
   (d) How many flags can be made if no two adjacent bars are the same color?

Solution. (a) We have a choice of 8 colours for each of the 4 bars, which yields

$$8^4 = 4096$$

different flags.

(b) If no two bars are to be the same, this leaves a choice of 8 colours for bar 1, 7 colours for bar 2, 6 colours for bar 3 and 5 colours for bar 4. Thus there are

$$8 \cdot 7 \cdot 6 \cdot 5 = 1680$$

different flags of this type. Note that this number can also be represented as ${}_8P_4 = \frac{8}{(8-4)!}$.

(c) We add the number of flags where all bars are different to the number of flags where the middle two bars must be the same. To count the latter, start by choosing 1 of 8 colours for the middle two bars, then choose 1 of 7 colours for the first bar, and 1 of 6 colours for the last bar. This gives

$$8 \cdot 7 \cdot 6 = 336$$

different flags where the middle two bars must have the same colour. Adding this to the number of flags in part (b) we get

$$336 + 1680 = 2016$$

different flags of the desired type.

(d) We have a choice of 8 colours for the first bar, leaving 7 colours for the second bar, then 7 colours for the third bar (different from bar 2 but can be the same as bar 1) and similarly 7 colours for the last bar. Thus there are

$$8 \cdot 7^3 = 2744$$

different flags of this type.

□

6. Eight students are registering for mathematics courses. There is room for 1 student in linear algebra, 3 students in calculus and 4 students in probability. How many different ways could these students register?

Solution. Choose 1 student for linear algebra, then choose 3 of the remaining 7 students for calculus, leaving the remaining 4 students to take probability. This gives

$$\binom{8}{1} \cdot \binom{7}{3} \cdot \binom{4}{4} = 8 \cdot 35 \cdot 1 = 280$$

ways these students could register.

Another way to obtain this is to count the number of words that can be made with the letters $ACCCPPPP$. Here A stands for linear algebra, C for calculus and P for probability. Order the students from 1 to 8 and assign each student the letter appearing in a given word, in the order that the letters appear. Note that there are

$$\frac{8!}{3! \cdot 4!} = \frac{40320}{144} = 280$$

words that can be made with $ACCCPPPP$.

This is the same counting used to find the number of ways in which $n = 8$ distinct objects (the students) can be partitioned into $k = 3$ subsets (the courses) with $n_1 = 1$ objects in the first (A), $n_2 = 3$ objects in the second (C), and $n_3 = 4$ objects in the third (P).

$$\binom{8}{1, 3, 4} = \frac{8!}{1! \cdot 3! \cdot 4!} = 280.$$

$\square$