MATH1550, Winter 2023 **Names:**
Seminar 11

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1. Let $b(x; n, \theta)$, be the binomial distribution where x is the number of "successes," n is the number of trials and θ is the probability of success at each trial. Compute the following:
   (a) $b(2; 5, \frac{1}{3})$.
   (b) $b(7; 10, \frac{1}{2})$.
   (c) $b(3; 4, \frac{1}{4})$.

   Solution. (a)
   $$b\left(2; 5, \frac{1}{3}\right) = \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = 10 \cdot \frac{1}{9} \cdot \frac{8}{27} = \frac{80}{243} \approx 0.3292.$$
   (b)
   $$b\left(7; 10, \frac{1}{2}\right) = \binom{10}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 = 120 \cdot \frac{1}{128} \cdot \frac{1}{8} = \frac{15}{128} \approx 0.1172.$$
   (c)
   $$b\left(3; 4, \frac{1}{4}\right) = \binom{4}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1 = 4 \cdot \frac{1}{64} \cdot \frac{3}{4} = \frac{3}{64} \approx 0.0469.$$
   □

2. A certain sports team has probability $\frac{2}{3}$ of winning a game whenever it plays.
   (a) Suppose 4 games are played, what is the probability that this team wins more than half of its games?
   (b) What is the probability they they lose all their games if they play 4 games?
   (c) Suppose they play 12 games. What is the probability they win more than 2 games?
   (d) If they play 15 games how many do they expect to win?

   Solution. We can model this situation with the binomial distribution.
   (a) The probability that they win more than half of their games if they play 4 games is
   $$b\left(3; 4, \frac{2}{3}\right) + b\left(4; 4, \frac{2}{3}\right) = \binom{4}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^1 + \binom{4}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^0 = \frac{32}{81} + \frac{16}{81} = \frac{16}{27} \approx 0.5926$$
   (b) The probability that they lose all 4 games is
   $$b\left(0; 4, \frac{2}{3}\right) = \binom{4}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^4 = \frac{1}{81} \approx 0.0123$$

(c)

$$\begin{aligned}
P(X > 2) &= 1 - P(X \le 2) \\
&= 1 - b\left(0; 12, \frac{2}{3}\right) - b\left(1; 12, \frac{2}{3}\right) - b\left(2; 12, \frac{2}{3}\right) \\
&= 1 - \binom{12}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{12} - \binom{12}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^{11} - \binom{12}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{10} \\
&= 1 - \frac{1}{3^{12}} - 12 \cdot \frac{2}{3^{12}} - 66 \cdot \frac{4}{3^{12}} \\
&= \frac{531152}{531441} \\
&\approx 0.9995
\end{aligned}$$

(d) For binomial distribution with $n = 15$ and $\theta = \frac{2}{3}$ we have

$$E(X) = n\theta = 15 \cdot \frac{2}{3} = 10.$$

$\square$

3. Find the least number of dice that must be thrown so that there is a better than 0.5 chance of rolling at least one 6. Assume these are fair 6-sided dice.

Solution. Let X be the number of 6's that appear in n rolls. The binomial distribution $b(k; n, \frac{1}{6})$, gives the the probability of rolling k 6's in n rolls. We are interesting in finding the smallest $n \in \mathbb{N}$ so that

$$0.5 < P(X \ge 1) = 1 - P(X = 0) = 1 - b\left(0; n, \frac{1}{6}\right) = 1 - \binom{n}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^n = 1 - \left(\frac{5}{6}\right)^n.$$

We can solve by trial and error to see that

$$\left(\frac{5}{6}\right)^3 = \frac{125}{216} > 0.5$$

whereas

$$\left(\frac{5}{6}\right)^4 = \frac{625}{1296} < 0.5$$

and thus $P(X \ge 1) > 0.5$ for $n = 4$, and this is the least such n.

Alternative: To avoid the trial and error part, note that

$$1 - \left(\frac{5}{6}\right)^n > 0.5 \quad \Rightarrow \quad \left(\frac{5}{6}\right)^n < 0.5 \quad \Rightarrow \quad n > \log_{\frac{5}{6}}(0.5) = 3.8,$$

(as $x < y$ implies $\log_a x > \log_a y$ for $0 < a < 1$) and hence $n = 4$.

$\square$

4. In an office building for a certain company, 8 employees will have to share an office. Each person has an equally likely chance of working from home versus working in the office. What is the minimum number of desks needed to be put in the office so that each person has a desk at least 90 percent of the time?

Solution. Letting X be the number of people working in the office, this situation can be modeled with the binomial distribution with $n = 8$ and $\theta = \frac{1}{2}$. The probability that k people will work in the office is given by $b(k; 8, \frac{1}{2})$ for $k = 0, \ldots, 8$.

Let d be the number of desks in the office, where $d = 0, \ldots, 8$. The probability that each person gets a desk is given by

$$P(X \leq d) = \sum_{k=0}^{d} b\left(k; 8, \frac{1}{2}\right) = \sum_{k=0}^{d} \binom{8}{k} \left(\frac{1}{2}\right)^8,$$

or by complement,

$$P(X \leq d) = 1 - P(X > d) = 1 - \sum_{k=d+1}^{8} b\left(k; 8, \frac{1}{2}\right) = 1 - \sum_{k=d+1}^{8} \binom{8}{k} \left(\frac{1}{2}\right)^8.$$

We want the least value for $d \in \{0, \ldots, 8\}$ so that $P(X \leq d) \geq 0.9$, which requires that $\sum_{k=d+1}^{8} \binom{8}{k} \left(\frac{1}{2}\right)^d < 0.1$. Note that

$$b\left(8; 8, \frac{1}{2}\right) = \binom{8}{8} \left(\frac{1}{2}\right)^8 = \frac{1}{256}$$

$$b\left(8; 7, \frac{1}{2}\right) = \binom{8}{7} \left(\frac{1}{2}\right)^8 = \frac{8}{256}$$

$$b\left(8; 6, \frac{1}{2}\right) = \binom{8}{6} \left(\frac{1}{2}\right)^8 = \frac{28}{256}$$

and so

$$\sum_{k=7}^{8} \binom{8}{k} \left(\frac{1}{2}\right)^8 = \frac{9}{256} \approx 0.0352 < 0.1$$

whereas

$$\sum_{k=6}^{8} \binom{8}{k} \left(\frac{1}{2}\right)^8 = \frac{37}{256} \approx 0.1445 > 0.1.$$

Therefore taking $d = 6$, yields $P(X \leq d) \geq 0.9$, and this is the least such d which ensure this. □

5. A certain basketball player is known to make free throws 72 percent of the time.
   (a) What is the probability that they will make their fifth free throw on their 8th shot?
   (b) What is the probability that they will make their fifth free throw on their 10th shot?
   (c) What is the expected number of free throws needed in order to make 5 baskets?

   Solution. The negative binomial distribution models this situation.

   (a) In this case $x = 8$, $\theta = 0.72$ and $k = 5$. Then

   $$b^*(8; 5, 0.72) = \binom{7}{4}(0.72)^5(0.28)^3 \approx 0.1487$$

   (b) In this case $x = 10$, $\theta = 0.72$ and $k = 5$. Then

   $$b^*(10; 5, 0.72) = \binom{9}{4}(0.72)^5(0.28)^5 \approx 0.04196$$

   (c)

   $$E(X) = \frac{k}{\theta} = \frac{5}{0.72} = \frac{125}{18} \approx 6.9444$$

$\square$

6. A discrete random variable X is said to have *Poisson distribution* if its probability distribution is given by

$$p(x;\lambda) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & x \in \{0,1,2,\dots,\} \\ 0 & \text{othewise} \end{cases}$$

for some $\lambda > 0$. The Poisson distribution $p(x,\lambda)$ may be used as an approximation for the binomial distribution $b(x;n,\theta)$, particularly when n is large and θ is small, by taking $\lambda = n\theta$.

(a) It is known that one million cars cross over a certain bridge every day. The probability that a car will get a flat tire is 0.00001. Write the expression for finding the probability that exactly 20 cars will have a flat tire using the binomial distribution.

(b) Use the Poisson distribution to approximate the probability in part (a).

Solution. (a) Using the binomial distribution, with $n = 1000000$ and $\theta = 0.00001$, we have

$$b(20; 1000000, 0.00001) = \binom{1000000}{20}(0.00001)^{20}(0.99999)^{9999970}$$

(b) To approximate part (a) with the Poisson distribution we take $\lambda = n\theta = (1000000)(0.00001) = 10$. Thus

$$p(20;10) = \frac{10^{20}e^{-10}}{20!} \approx 0.0019$$

$\square$