MATH1550, Winter 2023
Mini-Assignment 9 – Variance

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1. Suppose X is a discrete random variable whose probability distribution function $f(x)$ is given by the following table.

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{2}{10}$

Find the variance of X

Round to 4 decimal places if needed.

Solution. We first compute the expected values of X and X^2:

$$\begin{aligned} E(X) = \sum_x x f(x) &= (-2)\cdot\frac{1}{10} + (-1)\cdot\frac{3}{10} + 0\cdot\frac{3}{10} + 1\cdot\frac{1}{10} + 2\cdot\frac{2}{10} \\ &= -\frac{2}{10} - \frac{3}{10} + 0 + \frac{1}{10} + \frac{4}{10} = 0 \end{aligned}$$

$$\begin{aligned} E\left(X^2\right) = \sum_x x^2 f(x) &= (-2)^2\cdot\frac{1}{10} + (-1)^2\cdot\frac{3}{10} + 0^2\cdot\frac{3}{10} + 1^2\cdot\frac{1}{10} + 2^2\cdot\frac{2}{10} \\ &= \frac{4}{10} + \frac{3}{10} + 0 + \frac{1}{10} + \frac{8}{10} = \frac{16}{10} = \frac{8}{5} = 1.6 \end{aligned}$$

It follows that the variance of X is given by:

$$\operatorname{var}(X) = E\left(X^2\right) - [E(X)]^2 = \frac{8}{5} - 0^2 = \frac{8}{5} = 1.6.$$

□

2. Suppose X is a continuous random variable with probability density function

$$g(x) = \begin{cases} -\frac{x}{2} & -2 \le x \le 0 \\ 0 & \text{otherwiise} \end{cases}.$$

Find the variance of X.

Round to 4 decimal places if needed.

Solution. We first compute the expected values of X and X^2:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{-2} x \cdot 0\, dx + \int_{-2}^{0} x\left(-\frac{x}{2}\right)\, dx + \int_{0}^{\infty} x \cdot 0\, dx \\
&= 0 + \int_{-2}^{0}\left(-\frac{x^2}{2}\right)\, dx + 0 = \left(-\frac{x^3}{6}\right)\Big|_{-2}^{0} \\
&= \left(-\frac{0^3}{6}\right) - \left(-\frac{(-2)^3}{6}\right) = 0 - \left(-\frac{-8}{6}\right) = -\left(\frac{4}{3}\right) = -\frac{4}{3} \approx -1.3333
\end{aligned}$$

$$\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-\infty}^{-2} x^2 \cdot 0\, dx + \int_{-2}^{0} x^2\left(-\frac{x}{2}\right)\, dx + \int_{0}^{\infty} x^2 \cdot 0\, dx \\
&= 0 + \int_{-2}^{0}\left(-\frac{x^3}{2}\right)\, dx + 0 = \left(-\frac{x^4}{8}\right)\Big|_{-2}^{0} \\
&= \left(-\frac{0^4}{8}\right) - \left(-\frac{(-2)^4}{8}\right) = 0 - \left(-\frac{16}{8}\right) = -(-2) = 2
\end{aligned}$$

It follows that the variance of X is given by:

$$\operatorname{var}(X) = E\left(X^2\right) - [E(X)]^2 = 2 - \left[-\frac{4}{3}\right]^2 = 2 - \frac{16}{9} = \frac{2}{9} \approx 0.2222.$$

□

3. Suppose X is a discrete random variable whose probability distribution function $f(x)$ is given by the following table.

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{2}{10}$

Find the fourth moment of X.

Round to 4 decimal places if needed.

Solution. The fourth moment of X is the expected value of X^4:

$$\begin{aligned}
E\left(X^4\right) &= \sum_x x^4 f(x) = (-2)^4 \cdot \frac{1}{10} + (-1)^4 \cdot \frac{3}{10} + 0^4 \cdot \frac{3}{10} + 1^4 \cdot \frac{1}{10} + 2^4 \cdot \frac{2}{10} \\
&= \frac{16}{10} + \frac{3}{10} + 0 + \frac{1}{10} + \frac{16}{10} = \frac{36}{10} = 3.6
\end{aligned}$$

□

4. Let X be a random variable with a variance of 5. Find the variance of Y if $Y = 4 + 3X$.

Round to 4 decimal places if needed.

Solution.

$$\operatorname{var}(Y) = \operatorname{var}(4 + 3X) = 3^2 \operatorname{var}(X) = 45.$$

□

5. Suppose that a random variable Y has mean $\mu = 45$ and standard deviation $\sigma = 5$. Use Chebyshev's Inequality to find a minimum value for $P(35 < Y < 55)$.

Round to 4 decimal places if needed.

Solution. Here we go:

$$\begin{aligned} P(35 < Y < 55) &= P(35 - 45 < Y - 45 < 55 - 45) = P(-10 < Y - 45 < 10) \\ &= P(|Y - 45| < 10) = P(|Y - 45| < 2 \cdot 5) = P(|Y - \mu| < 2\sigma) \end{aligned}$$

$$\text{By Chebyshev:} \quad P(|Y - \mu| < 2\sigma) \geq 1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

Thus $P(35 < Y < 55)$ must be at least $\frac{3}{4} = 0.75$.

□

6. Suppose X is a discrete random variable whose probability distribution function $f(x)$ is given by the following table.

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{2}{10}$

What is the probability that X lies within 1 standard deviation of its mean?

Round to 4 decimal places if needed.

Solution. This is the same probability distribution function that we saw in question **1**, so X has mean $\mu = E(X) = 0$ and variance $\sigma^2 = \text{var}(X) = 1.6$, and thus standard deviation $\sigma = \sqrt{1.6} \approx 1.2649$. X takes on three possible values between $\mu - \sigma = 0 - 1.2649 = -1.2649$ and $\mu + \sigma = 0 + 1.2649 = 1.2649$, namely -1, 0, and 1. Thus

$$\begin{aligned} P(|X - \mu| < \sigma) &= P(|X - 0| < 1.2649) = P(X = -1) + P(X = 0) + P(X = 1) \\ &= f(-1) + f(0) + f(1) = \frac{3}{10} + \frac{3}{10} + \frac{1}{10} = \frac{7}{10} = 0.7 \end{aligned}$$

□