MATH1550, Winter 2023
Mini-Assignment 7 – More Joint Distributions

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1. The joint probability density function of the continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} e^{-x}y^{-2} & x \geq 0 \text{ and } y \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find $P(0 \leq X \leq \ln(2),\, 0 \leq Y \leq 2)$.

Round to 4 decimal places if needed.

Solution.

$$\begin{aligned} P(0 \leq X \leq \ln(2),\, 0 \leq Y \leq 2) &= \int_0^{\ln 2} \int_1^2 e^{-x}y^{-2}\, dy\, dx \\ &= \int_0^{\ln 2} -e^{-x}y^{-1}\Big|_1^2\, dx \\ &= \int_0^{\ln 2} \frac{e^{-x}}{2} dx \\ &= -\frac{e^{-x}}{2}\Big|_0^{\ln 2} \\ &= \left(-\frac{1}{4}\right) - \left(-\frac{1}{2}\right) \\ &= \frac{1}{4} \end{aligned}$$

□

2. The joint probability distribution function $f(x,y)$ of the discrete random variables X and Y is given by the following table:

y \ x	0	1	2	3
2			0.1	0.1
3	0.1		0.1	
4		0.1	0.1	0.1
5	0.1	0.1	0.1	

Let $h(y)$ be the marginal probability distribution function of Y. What is $h(3)$?

Round to 4 decimal places if needed.

Solution.

$$h(3) = \sum_{x=0}^{3} f(x,3) = 0.1 + 0 + 0.1 + 0 = 0.2$$

□

3. Is the following function a valid joint probability density function?

$$f(x,y) = \begin{cases} \frac{1}{4} & 0 \le x \le 2, 0 \le y < 1 \\ \frac{1}{6} & 0 \le x \le 1, 1 \le y \le 2 \\ \frac{1}{3} & 1 < x \le 2, 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution. We can see that $f(x,y) \ge 0$ for all $x, y \in \mathbb{R}$, so it suffices to check whether $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx &= \int_0^2\int_0^1 \frac{1}{4}\, dy\, dx + \int_0^1\int_1^2 \frac{1}{6}\, dy\, dx + \int_1^2\int_1^2 \frac{1}{3}\, dy\, dx \\
&= \int_0^2 \frac{y}{4}\Big|_0^1\, dx + \int_0^1 \frac{y}{6}\Big|_1^2\, dx + \int_1^2 \frac{y}{3}\Big|_1^2\, dx \\
&= \int_0^2 \frac{1}{4}\, dx + \int_0^1 \frac{1}{6}\, dx + \int_1^2 \frac{1}{3}\, dx \\
&= \frac{x}{4}\Big|_0^2 + \frac{x}{6}\Big|_0^1 + \frac{x}{3}\Big|_1^2 \\
&= \frac{1}{2} + \frac{1}{6} + \frac{1}{3} \\
&= 1
\end{aligned}$$

Therefore $f(x,y)$ is a valid joint density function. □

4. The joint probability density function of the continuous random variables X and Y is

$$f(x,y) = \begin{cases} x - y & 0 \le x \le 1, -1 \le y \le 0 \\ 0 & \text{otherwise} \end{cases}.$$

Find $g(x)$, the marginal probability density function of X?

A: $g(x) = \begin{cases} -y - \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$, B: $g(x) = \begin{cases} -x - \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$, C: $g(x) = \begin{cases} \frac{1}{2} - y & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$,

D: $g(x) = \begin{cases} \frac{1}{2} - x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$, E: $g(x) = \begin{cases} x - \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$, F: $g(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$,

G: $g(x) = \begin{cases} xy - \frac{y^2}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$, H: $g(x) = \begin{cases} \frac{x^2}{2} - xy & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$, I: Neither

Solution. For $0 \le x \le 1$,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_{-1}^{0} x - y\, dy = xy - \frac{y^2}{2}\Big|_{-1}^{0} = x + \frac{1}{2}$$

It follows that

$$g(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

□

5. The joint probability distribution function $f(x, y)$ of the discrete random variables X and Y is given by the following table:

y \ x	0	1	2	3
2			0.1	0.1
3	0.1		0.1	
4		0.1	0.1	0.1
5	0.1	0.1	0.1	

Let $f(x|y)$ be the conditional distribution function for X given $Y = y$. Find $f(2|4)$.

Round to 4 decimal places if needed.

Solution. First note that

$$h(4) = \sum_{x=0}^{3} f(x, 4) = 0 + 0.1 + 0.1 + 0.1 = 0.3.$$

Then

$$f(2|4) = \frac{f(2,4)}{h(4)} = \frac{0.1}{0.3} = \frac{1}{3}.$$

□

6. The joint probability density function of the continuous random variables X and Y is

$$f(x, y) = \begin{cases} x - y & 0 \le x \le 1, -1 \le y \le 0 \\ 0 & \text{otherwise} \end{cases}.$$

Let $f(y|x)$ be the conditional density function for Y given $X = x$. Find $P(-0.5 \le Y \le 0|X = 0.5)$.

Round to 4 decimal places if needed.

Solution. We have from above that

$$g(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Then, for $0 \le x \le 1, -1 \le y \le 0$,

$$f(y|x) = \frac{x - y}{x + \frac{1}{2}} \quad \Rightarrow \quad f(y|0.5) = \frac{1}{2} - y.$$

Thus

$$P(-0.5 \leq Y \leq 0|X = 0.5) = \int_{-0.5}^{0} f(y|0.5)\ dy = \int_{-0.5}^{0} \frac{1}{2} - y\ dy = \frac{y}{2} - \frac{y^2}{2}\Big|_{-0.5}^{0} = \frac{3}{8}$$

□