MATH1550, Winter 2023
Mini-Assignment 6 – Cumulative and Joint Distributions

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1. Suppose

$$F(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ \sqrt{x} & x \in (0,1) \\ 1 & x \in [1, \infty) \end{cases}$$

is the cumulative probability function of the random variable X. What is the probability density function $f(x)$ of X?

A: $f(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ x^{\frac{1}{2}} & x \in (0,1) \\ 1 & x \in [1, \infty) \end{cases}$, B: $f(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ \frac{1}{\sqrt{x}} & x \in (0,1) \\ 1 & x \in [1, \infty) \end{cases}$, C: $f(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ \frac{2x^{\frac{3}{2}}}{3} & x \in (0,1) \\ x & x \in [1, \infty) \end{cases}$

D: $f(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ \frac{1}{2\sqrt{x}} & x \in (0,1) \\ 1 & x \in [1, \infty) \end{cases}$, E: $f(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ \frac{1}{2\sqrt{x}} & x \in (0,1) \\ x & x \in [1, \infty) \end{cases}$, F: $f(x) = \begin{cases} \frac{1}{2\sqrt{x}} & x \in (0,1) \\ 0 & \text{else} \end{cases}$

G: $f(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ \frac{2}{\sqrt{x}} & x \in (0,1) \\ 1 & x \in [1, \infty) \end{cases}$, H: $f(x) = \begin{cases} \frac{2x^{\frac{3}{2}}}{3} & x \in (0,1) \\ 0 & \text{else} \end{cases}$, I: $f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{2\sqrt{x}} & x > 0 \end{cases}$

J: $f(x) = \begin{cases} \sqrt{x} & x \in (0,1) \\ 0 & \text{else} \end{cases}$, K: $f(x) = \begin{cases} \frac{2}{\sqrt{x}} & x \in (0,1) \\ 0 & \text{else} \end{cases}$, L: Neither

Solution. Take the derivative of each piece of $F(x)$ to obtain $f(x)$:

$$f(x) = \begin{cases} \frac{1}{2\sqrt{x}} & x \in (0,1) \\ 0 & \text{else} \end{cases}$$

□

2. The random variable W has probability density function

$$g(w) = \begin{cases} 0 & w < 1 \\ \frac{1}{w^2} & w \geq 1 \end{cases}.$$

What is the cumulative probability function $G(w)$ of W?

A: $G(w) = \begin{cases} 0 & w < 1 \\ \frac{1}{w^2} & w \geq 1 \end{cases}$, B: $G(w) = \begin{cases} 0 & w < 1 \\ 1 + \frac{1}{w} & w \geq 1 \end{cases}$, C: $G(w) = \begin{cases} 0 & w < 1 \\ \frac{1}{w} & w \geq 1 \end{cases}$

D: $G(w) = \begin{cases} 0 & w < 0 \\ 1 - \frac{1}{w^2} & w \in (0,1) \\ 1 & w \geq 1 \end{cases}$, E: $G(w) = \begin{cases} 0 & w < 1 \\ 1 - \frac{1}{w} & w \in (0,1) \\ 1 & w \geq 1 \end{cases}$, F: $G(w) = \begin{cases} 0 & w < 1 \\ 1 - \frac{2}{w^3} & w \geq 1 \end{cases}$

$$\text{G: } G(w) = \begin{cases} 0 & w < 1 \\ -\frac{1}{w} & w \geq 1 \end{cases}, \quad \text{H: } G(w) = \begin{cases} 0 & w < 1 \\ 1 - \frac{1}{w} & w \geq 1 \end{cases}, \quad \text{I: } G(w) = \begin{cases} 0 & w < 1 \\ \frac{2}{w^3} & w \geq 1 \end{cases}$$

$$\text{J: } G(w) = \begin{cases} 0 & w < 0 \\ 1 - \frac{1}{w} & w \in (0,1) \\ 1 & w \geq 1 \end{cases}, \quad \text{K: } G(w) = \begin{cases} 0 & w < 1 \\ -\frac{1}{w} & w \in (0,1) \\ 1 & w \geq 1 \end{cases}, \quad \text{L: Neither}$$

Solution. For $w < 1$, $G(w) = 0$. For $w \geq 1$,

$$G(w) = \int_{-\infty}^{w} g(t)\, dt = \int_{1}^{w} \frac{1}{t^2}\, dt = -\frac{1}{t}\Big|_{1}^{w} = 1 - \frac{1}{w}.$$

Thus

$$G(w) = \begin{cases} 0 & w < 1 \\ 1 - \frac{1}{w} & w \geq 1 \end{cases}$$

□

3. The random variable T describes when an atom of a certain radioactive element decays after start time $t = 0$. T has the probability density function

$$h(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases}.$$

Find the half-life of the element in question, *i.e.* the value of t such that $P(T \leq t) = \frac{1}{2}$.

Round to 4 decimal places if necessary.

Solution.

$$\frac{1}{2} = P(T \leq t) = \int_{-\infty}^{t} h(x)\, dx = \int_{0}^{t} e^{-x}\, dx = -e^{-x}\Big|_{0}^{t} = 1 - e^{-t},$$

thus

$$\frac{1}{2} = 1 - e^{-t} \quad \Rightarrow \quad e^{-t} = \frac{1}{2} \quad \Rightarrow \quad e^{t} = 2 \quad \Rightarrow \quad t = \ln 2 \approx 0.6931$$

□

4. Is

$$G(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ 1 - xe^{-x} & x \in (0, \infty) \end{cases}$$

a valid cumulative probability function for some random variable X?

Solution. Note that $G(0) = 1$ and $G(1) = 1 - \frac{1}{e} < 1$. This violates the property that $G(a) \leq G(b)$ for $a < b$ when G is a cumulative distribution function (i.e. cumulative probability should only increase). Thus G is not a valid cumulative probability distribution function.

□

5. Two fair three-sided dice, one blue and one red, each have faces numbered 1, 2 and 3. The pair of dice are rolled twice (i.e. both dice are rolled once, then both dice are rolled again). Let X be the number of times, out the 2 rolls, in which the sum of the two dice is an even number. Let Y be the number of times, out of the 2 rolls, in which the blue die comes up with an odd-number. If $f(x, y)$ is the joint probability distribution of X and Y, what is $f(0, 2)$?

Round to 4 decimal places if necessary.

Solution. To have $X = 0$ and $Y = 2$ means that both rolls had an odd sum, and in both rolls the blue die had an odd number. This means the blue die is either ⚀ or ⚂ and the red die is ⚁ . There are $3 \cdot 3 = 9$ different pairs that can come up on a single roll, and hence $9 \cdot 9 = 81$ different ways the two rolls could come out. There are 4 "successful" outcomes in this case:

(⚀⚁, ⚀⚁), (⚀⚁, ⚂⚁), (⚂⚁, ⚀⚁), (⚂⚁, ⚂⚁).

Therefore $f(0, 2) = \frac{4}{81}$.

As an exercise, show that the joint probability distribution is

		x = 0	1	2
y	0	$\frac{4}{81}$	$\frac{4}{81}$	$\frac{1}{81}$
	1	$\frac{8}{81}$	$\frac{20}{81}$	$\frac{8}{81}$
	2	$\frac{4}{81}$	$\frac{16}{81}$	$\frac{16}{81}$

□

6. The discrete random variables U and W have the joint probability distribution $g(u, w)$ given by the following table:

		u = −1	0	1
w	1	0.1	0	0.2
	2	0.1	0.2	0
	3	0.2	0.1	0.1

Let $G(u, w)$ be the cumulative probability distribution function of U and W. Compute $G(2, 1)$.

Round to 4 decimal places if necessary.

Solution.

$$G(2, 1) = P(U \leq 2, W \leq 1) = g(-1, 1) + g(0, 1) + g(1, 1) = 0.1 + 0 + 0.2 = 0.3.$$

□