MATH1550, Winter 2023
Mini-Assignment 5 – Continuous Distributions

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1. Which of the following functions are allowable as a probability density function for some continuous random variable?

$$\text{A: } f_1(x) = \begin{cases} 3x^2 & -1 \le x \le 0 \\ 0 & \text{else} \end{cases} \qquad \text{B: } f_2(x) = \begin{cases} 5(1-x^4)^4 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$\text{C: } f_3(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \le 10 \end{cases} \qquad \text{D: } f_4(x) = \begin{cases} \frac{3}{2}(8x^2 - 6x + 1) & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$\text{E: } f_5(x) = \begin{cases} 1 - x^2 & -1 < x < 1 \\ 0 & \text{else} \end{cases} \qquad \text{F: Neither}$$

Solution. For each function we must check whether $f(x) \ge 0$ for all $x \in \mathbb{R}$, and if $\int_{-\infty}^{\infty} f(x) = 1$. One can see fairly easily by inspection that $f_1(x), f_2(x), f_3(x), f_5(x) \ge 0$ for all $x \in \mathbb{R}$, so that leaves $f_4(x)$. Solving $8x^2 - 6x + 1 = 0$ with the quadratic formula we have

$$x = \frac{6 \pm \sqrt{4}}{16} = \quad \Rightarrow \quad x = \frac{1}{4} \text{ or } x = \frac{1}{2}.$$

Since $8x^2 - 6x + 1$ is a parabola opening upwards, we see that $8x^2 - 6x + 1 < 0$, and hence $\frac{3}{2}(8x^2 - 6x + 1) < 0$ on the interval $(\frac{1}{4}, \frac{1}{2})$. This rules out $f_4(x)$ as a valid probability density function.

Next we check the second property.

$$\int_{-\infty}^{\infty} f_1(x)\,dx = \int_{-1}^{0} 3x^2\,dx = x^3 \Big|_{-1}^{0} = 0 - (-1) = 1.$$

$$\begin{aligned} \int_{-\infty}^{\infty} f_2(x)\,dx &= \int_0^1 5(1-x^4)^4\,dx \\ &= 5\int_0^1 1 - 4x^4 + 6x^8 - 4x^{12} + x^{16}\,dx \\ &= 5\left(x - \frac{4x^5}{5} + \frac{2x^9}{3} - \frac{4x^{13}}{13} + \frac{x^{17}}{17}\right)\Big|_0^1 \\ &= \frac{2048}{663} \\ &\ne 1. \end{aligned}$$

$$\int_{-\infty}^{\infty} f_3(x)\,dx = \int_{10}^{\infty} \frac{10}{x^2}\,dx = -\frac{10}{x}\Big|_{10}^{\infty} = 0 - (-1) = 1.$$

$$\int_{-\infty}^{\infty} f_5(x)\,dx = \int_{-1}^{1} 1 - x^2\,dx = x - \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3} \ne 1.$$

Therefore only $f_1(x)$ and $f_3(x)$ are valid probability density functions. □

2. Consider the function

$$f(x) = \begin{cases} k(2x - x^2) & 0 < x < \frac{5}{2} \\ 0 & \text{else} \end{cases}$$

If possible, find a value for $k \in \mathbb{R}$ which makes this a valid probability density function.

A: $k = 1$, B: $k = -1$ C: $k = \frac{24}{25}$ D: $k = \frac{25}{24}$

E: $k = 24$ F: $k = 25$ G: $k = 0$ H: No such k exists.

Solution. Note that $f(x) = k(2x - x^2) = kx(2 - x)$ is zero when $x = 0$ and $x - 2$. Since this is a parabola opening downwards, it follows that $f(x) > 0$ for $x \in (0, 2)$ and $f(x) < 0$ for $x \in (2, \frac{5}{2})$. Since the function takes both positive and negative values, there can be no $k \in \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in \mathbb{R}$. □

3. The cumulative distribution function for a continuous random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \leq x \leq 1 \\ 1 - \frac{1}{2}(2 - x)^2 & 1 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

Find $P(0.5 < X < 1.5)$

Round to 4 decimal places if needed.

Solution.

$$P(0.5 < X < 1.5) = F(1.5) - F(0.5) = \frac{7}{8} - \frac{1}{8} = \frac{3}{4}.$$

□

4. The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

Find $P(0.5 < X < 1.5)$

Round to 4 decimal places if needed.

Solution.

$$\begin{aligned} P(0.5 < X < 1.5) &= \int_{0.5}^{1.5} f(x)\, dx \\ &= \int_{0.5}^{1} x\, dx + \int_{1}^{1.5} 2 - x\, dx \\ &= \left[\frac{x^2}{2}\right]_{0.5}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{1.5} \\ &= \left(\frac{1}{2} - \frac{1}{8}\right) + \left(\frac{15}{8} - \frac{3}{2}\right) \\ &= \frac{3}{4} \end{aligned}$$

□

5. The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{1}{3} & 0 < x < 1 \\ \frac{1}{3} & 2 < x < 4 \\ 0 & \text{else} \end{cases}$$

and its cumulative distribution function is given (partially) by

$$F(x) = \begin{cases} 0 & x \leq 0 \\ R_1 & 0 < x < 1 \\ R_2 & 1 \leq x \leq 2 \\ R_3 & 2 < x < 3 \\ R_4 & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

Find the rule for R_2.

A: $R_2 = 0$ B: $R_2 = \dfrac{1}{3}$ C: $R_2 = \dfrac{2}{3}$ D: $R_2 = \dfrac{x}{3}$ E: $R_2 = \dfrac{2x}{3}$ F: Neither

Solution. For $0 < x < 1$:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x \frac{1}{3}\,dt = \frac{t}{3}\bigg|_0^x = \frac{x}{3}.$$

For $1 \leq x \leq 2$:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^1 \frac{1}{3}\,dt + \int_1^x 0\,dt = \frac{t}{3}\bigg|_0^1 = \frac{1}{3}.$$

For $2 < x < 4$:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^1 \frac{1}{3}\,dt + \int_1^2 0\,dt + \int_2^x \frac{1}{3}\,dt = \frac{t}{3}\bigg|_0^1 + \frac{t}{3}\bigg|_2^x = \frac{x}{3} - \frac{1}{3}.$$

Thus

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{3} & 0 < x < 1 \\ \frac{1}{3} & 1 \leq x \leq 2 \\ \frac{x}{3} - \frac{1}{3} & 2 < x < 3 \\ \frac{x}{3} - \frac{1}{3} & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

and we see that $R_2 = \frac{1}{3}$. □

6. The shelf life (in days) of a certain packaged food item is a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{20000}{(x+100)^3} & x > 0 \\ 0 & \text{else} \end{cases}$$

What is the probability that a given package will last at least 100 days?

Round to 4 decimal places if needed.

Solution.

$$P(X \geq 100) = \int_{100}^{\infty} \frac{20000}{(x+100)^3}\ dx = -\frac{10000}{(x+100)^2}\Big|_{100}^{\infty} = 0 - \left(-\frac{10000}{40000}\right) = \frac{1}{4}.$$

□