MATH1550, Winter 2023
Mini-Assignment 4 – Discrete Distributions

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1. The *face cards* in a standard 52-card deck are the K's, Q's, and J's. A seven card hand is chosen randomly, all seven at once, from the deck. The discrete random variable Y counts how many face cards there are in the hand. What is the probability distribution function of Y ?

$$\text{A: } f(y) = \frac{\binom{4}{y}\binom{3}{3}\binom{52}{7-y}}{\binom{52}{7}}, \quad \text{B: } f(y) = \frac{\binom{13}{3}\binom{12}{7}}{\binom{52}{7}}, \quad \text{C: } f(y) = \frac{\binom{3}{1}\binom{4}{y}\binom{40}{7-y}}{\binom{52}{7}}, \quad \text{D: } f(y) = \frac{\binom{12}{7}\binom{40}{y}}{\binom{52}{7}},$$

$$\text{E: } f(y) = \frac{\binom{12}{y}\binom{40}{7-y}}{\binom{52}{7}}, \quad \text{F: } f(y) = \frac{\binom{13}{7}\binom{4}{3}\binom{7}{y}}{\binom{52}{7}}, \quad \text{G: } f(y) = \frac{\binom{4}{y}\binom{4}{y-1}\binom{4}{y-2}\binom{40}{7-y}}{\binom{52}{7}}, \quad \text{H: Neither}$$

Solution. The range of Y is $\{0, 1, 2, 3, 4, 5, 6, 7\}$. The probability distribution for Y is

$$P(Y = y) = \frac{\binom{12}{y}\binom{40}{7-y}}{\binom{52}{7}}$$

□

2. Find the value of k, if one exists, such that $f(x) = \frac{x}{k}$ is a valid probability distribution function for a discrete random variable X which can take on the values 0, 1, 2, 3, or 4.

Round to 4 decimal places if needed.

Solution. We require that

$$1 = \sum_{x=0}^{4} \frac{x}{k} = \frac{0+1+2+3+4}{k} = \frac{10}{k}$$

which forces $k = 10$. Since $\frac{x}{10} \geq 0$ for each $x \in \{0, 1, 2, 3, 4\}$, we see that $f(x) = \frac{x}{10}$ is a valid probability distribution for X. □

3. A fair standard die is rolled twice. The discrete random variable Y is given by the number that came up on the first roll minus the number that came up on the second roll. What is the cumulative distribution function of Y?

$$\text{A: } F(y) = \begin{cases} 0 & \text{for } y < -5 \\ \frac{1}{36} & \text{for } -5 \le y < -4 \\ \frac{3}{36} & \text{for } -4 \le y < -3 \\ \frac{6}{36} & \text{for } -3 \le y < -2 \\ \frac{10}{36} & \text{for } -2 \le y < -1 \\ \frac{15}{36} & \text{for } -1 \le y < 0 \\ \frac{21}{36} & \text{for } 0 \le y < 1 \\ \frac{26}{36} & \text{for } 1 \le y < 2 \\ \frac{30}{36} & \text{for } 2 \le y < 3 \\ \frac{33}{36} & \text{for } 3 \le y < 4 \\ \frac{35}{36} & \text{for } 4 \le y < 5 \\ 1 & \text{for } y \ge 5 \end{cases} \quad \text{B: } F(y) = \begin{cases} \frac{1}{36} & \text{for } y = -5 \\ \frac{2}{36} & \text{for } y = -4 \\ \frac{3}{36} & \text{for } y = -3 \\ \frac{4}{36} & \text{for } y = -2 \\ \frac{5}{36} & \text{for } y = -1 \\ \frac{6}{36} & \text{for } y = 0 \\ \frac{5}{36} & \text{for } y = 1 \\ \frac{4}{36} & \text{for } y = 2 \\ \frac{3}{36} & \text{for } y = 3 \\ \frac{2}{36} & \text{for } y = 4 \\ \frac{1}{36} & \text{for } y = 5 \end{cases}, \quad \text{C: } F(y) = \begin{cases} \frac{1}{36} & \text{for } y = -5 \\ \frac{3}{36} & \text{for } y = -4 \\ \frac{6}{36} & \text{for } y = -3 \\ \frac{10}{36} & \text{for } y = -2 \\ \frac{15}{36} & \text{for } y = -1 \\ \frac{21}{36} & \text{for } y = 0 \\ \frac{26}{36} & \text{for } y = 1 \\ \frac{30}{36} & \text{for } y = 2 \\ \frac{33}{36} & \text{for } y = 3 \\ \frac{35}{36} & \text{for } y = 4 \\ 1 & \text{for } y = 5 \end{cases}$$

$$\text{D: } F(y) = \begin{cases} 0 & \text{for } y \le -5 \\ \frac{1}{36} & \text{for } -5 < y \le -4 \\ \frac{3}{36} & \text{for } -4 < y \le -3 \\ \frac{6}{36} & \text{for } -3 < y \le -2 \\ \frac{10}{36} & \text{for } -2 < y \le -1 \\ \frac{15}{36} & \text{for } -1 < y \le 0 \\ \frac{21}{36} & \text{for } 0 < y \le 1 \\ \frac{26}{36} & \text{for } 1 < y \le 2 \\ \frac{30}{36} & \text{for } 2 < y \le 3 \\ \frac{33}{36} & \text{for } 3 < y \le 4 \\ \frac{35}{36} & \text{for } 4 < y \le 5 \\ 1 & \text{for } y > 5 \end{cases}, \quad \text{E: } F(y) = \begin{cases} 0 & \text{for } y < -5 \\ \frac{1}{36} & \text{for } -5 \le y < -4 \\ \frac{2}{36} & \text{for } -4 \le y < -3 \\ \frac{3}{36} & \text{for } -3 \le y < -2 \\ \frac{4}{36} & \text{for } -2 \le y < -1 \\ \frac{5}{36} & \text{for } -1 \le y < 0 \\ \frac{6}{36} & \text{for } 0 \le y < 1 \\ \frac{5}{36} & \text{for } 1 \le y < 2 \\ \frac{4}{36} & \text{for } 2 \le y < 3 \\ \frac{3}{36} & \text{for } 3 \le y < 4 \\ \frac{2}{36} & \text{for } 4 \le y < 5 \\ \frac{1}{36} & \text{for } y \ge 5 \end{cases}, \quad \text{F: Neither}$$

Solution. The range of Y is $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. The probability distribution for Y is as follows

y	-5	-4	-3	-2	-1	0	1	2	3	4	5
$P(Y = y)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

or

$$P(Y = y) = \frac{6 - |y|}{36}$$

The cumulative distribution is therefore

$$F(y) = P(Y \leq y) = \begin{cases} 0 & \text{for } y < -5 \\ \frac{1}{36} & \text{for } -5 \leq y < -4 \\ \frac{3}{36} & \text{for } -4 \leq y < -3 \\ \frac{6}{36} & \text{for } -3 \leq y < -2 \\ \frac{10}{36} & \text{for } -2 \leq y < -1 \\ \frac{15}{36} & \text{for } -1 \leq y < 0 \\ \frac{21}{36} & \text{for } 0 \leq y < 1 \\ \frac{26}{36} & \text{for } 1 \leq y < 2 \\ \frac{30}{36} & \text{for } 2 \leq y < 3 \\ \frac{33}{36} & \text{for } 3 \leq y < 4 \\ \frac{35}{36} & \text{for } 4 \leq y < 5 \\ 1 & \text{for } y \geq 5 \end{cases}$$

□

4. Find the value of k, if one exists, such that $F(x) = \frac{x+2}{k}$ is a valid cumulative distribution function for a discrete random variable X which can take on the values -2, -1, 0, 1, or 2.

   Round to 4 decimal places if needed.

   Solution. We require that

$$1 = F(2) = \frac{4}{k}$$

   which implies that $k = 4$. Note that $\frac{x+2}{4} \geq 0$ for each $x \in \{-2, -1, 0, 1, 2\}$. □

5. A fair coin is tossed twice until it comes up with the same face on two consecutive tosses. The discrete random variable X counts the number of tosses that are made. What is the probability distribution function of X?

$$\text{A: } P(X = x) = \frac{1}{2}, \quad \text{B: } P(X = x) = \left(\frac{1}{2}\right)^x, \quad \text{C: } P(X = x) = \left(\frac{1}{2}\right)^{x-1}, \quad \text{D: } P(X = x) = \left(\frac{1}{2}\right)^{x+1},$$

$$\text{E: } P(X = x) = \frac{1}{2x}, \quad \text{F: } P(X = x) = \frac{\binom{x}{2}}{2^x}, \quad \text{G: } P(X = x) = \frac{\binom{x}{2}}{2^{x+1}}, \quad \text{H: Neither}$$

   Solution. The range of X is the infinite set $\mathbb{N} \setminus \{1\} = \{2, 3, 4, \ldots\}$. The probability distribution for X is given by

$$P(X = x) = \left(\frac{1}{2}\right)^{x-1}.$$

   Explanation: Use the multiplication rule for independent events (an event being a single coin toss). To get a match after x tosses, the first $x - 1$ tosses must alternate between heads and tails and the xth toss must match the $(x - 1)$th toss. The first toss doesn't matter, but the ith toss must be different from the $(i - 1)$th toss for $2 \leq i < x - 1$; each occurs with probability $\frac{1}{2}$ and so the product is $(\frac{1}{2})^{x-2}$. The xth toss matching the $(x - 1)$th toss occurs with probability $\frac{1}{2}$. Multiplying these two we get $(\frac{1}{2})^{x-1}$. □

6. Suppose the random variable Z has the following cumulative distribution function:

$$F(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{8} & 0 \leq z < 1 \\ \frac{3}{16} & 1 \leq z < 2 \\ \frac{3}{8} & 2 \leq z < 4 \\ \frac{11}{16} & 4 \leq z < 5 \\ \frac{23}{32} & 5 \leq z < 6 \\ \frac{7}{8} & 6 \leq z < 7 \\ 1 & 7 \leq z \end{cases}$$

The probability distribution function of Z, is

$$f(z) = \begin{cases} 0 & z < 0 \\ A & z = 0 \\ \frac{1}{16} & z = 1 \\ B & z = 2 \\ \frac{5}{16} & z = 4 \\ C & z = 5 \\ \frac{5}{32} & z = 6 \end{cases}$$

What is $A + B + C$?

Round to 4 decimal places if needed.

Solution. If $f(z)$ is the probability distribution function for Z then $f(z_i) = F(z_i) = F(z_{i-1})$ (where the z_i are the range values for Z in order). This yields that

$$f(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{8} & z = 0 \\ \frac{1}{16} & z = 1 \\ \frac{3}{16} & z = 2 \\ \frac{5}{16} & z = 4 \\ \frac{1}{32} & z = 5 \\ \frac{5}{32} & z = 6 \end{cases}$$

Therefore

$$A + B + C = \frac{1}{8} + \frac{3}{16} + \frac{1}{32} = \frac{11}{32}$$

□