MATH1550, Winter 2023
Mini-Assignment 2 – Basic Probability

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1. A coin with a head on one side and a tail on the other is tossed, followed by a six-sided die with faces marked 1 through 6 being rolled. If each outcome is equally likely, what is the probability of each outcome?

   Solution. Since there are 2 outcomes for the coin toss and 6 outcomes for the die roll, there are 12 possible outcomes in total, namely

   $$\{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}.$$

   Each is equally likely and therefore has a probability of $\frac{1}{12} \approx 0.0833$. □

2. Events A and B in some sample space each have probability 0.5 of occurring, while the probability that both occur (simultaneously) is 0.25. What is the probability that neither of A or B occurs?

   Solution. The event that neither A or B occurs is $(A \cup B)'$. Then

   $$\begin{aligned} P((A \cup B)') &= 1 - P(A \cup B) \\ &= 1 - (P(A) + P(B) - P(A \cap B) \\ &= 1 - (0.5) - (0.5) + (0.25) \\ &= 0.25 \end{aligned}$$

   □

3. A five-card hand is drawn from a standard 52-card deck all at once, so order doesn't matter. What is the probability that the hand consists entirely of face cards (K, Q, and J are the ones called *face cards*)?

   Solution. Since there are 12 face cards, then number of different 5-card hands of all face cards (not counting order) is $\binom{12}{5}$. The total number of different 5 card hands (not counting order) is $\binom{52}{5}$. Since each 5-card hand has an equally chance of being drawn the probability of drawing 5 face cards is

   $$\frac{\binom{12}{5}}{\binom{52}{5}} \approx 0.0003.$$

   □

4. For events A, B and C we have that $P(A) = 0.24$, $P(B) = 0.26$, $P(A \cap B) = 0.05$, $P(A \cap C) = 0.05$, $P(B \cap C) = 0.05$, $P(A \cap B \cap C) = 0.02$ and $P((A \cup B \cup C)') = 0.33$. Find $P(C)$.

   Solution. By the inclusion-exclusion principle,

   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C),$$

   so

   $$P(C) = P(A \cup B \cup C) - P(A) - P(B) + P(A \cap B) + P(A \cap C) + P(B \cap C) - P(A \cap B \cap C).$$

   Since

   $$P(A \cup B \cup C) = 1 - P((A \cup B \cup C)')$$

we have

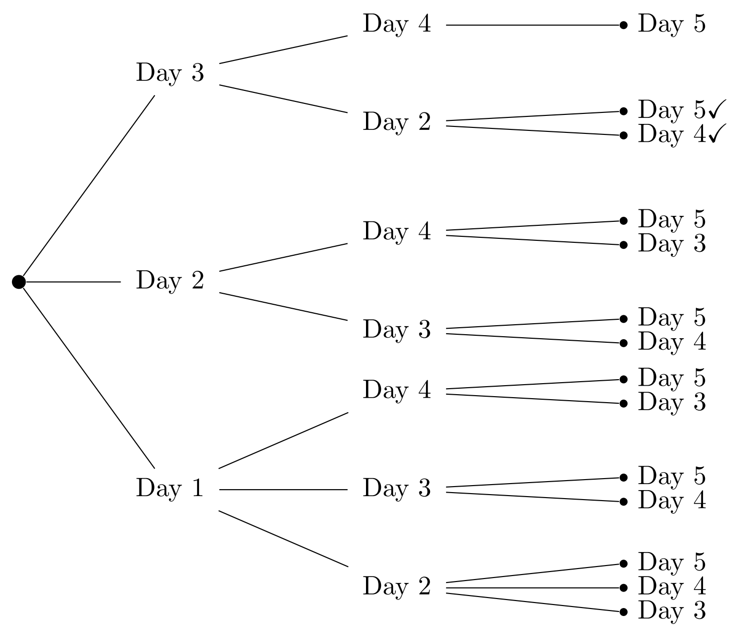
$$P(C) = 0.67 - 0.24 - 0.26 + 0.05 + 0.05 + 0.05 - 0.02 = 0.3.$$

□

5. Meredith orders three items from an online retailer. The first item will arrive with equal likelihood in one of the first through third days of February. The second will arrive with equal likelihood in one of the second through fourth days of February, and the third will arrive with equal likelihood in one of the third through fifth days of February. If at most one item can be delivered in a given day, what is the probability that the first item is not the first to be delivered?

   Solution. Authors' note: The wording on this question (inspired by real events) was not what we had initially intended, but it still makes for a nice problem! Below are some ways to solve.

   We can see this with a tree diagram (or simply listing the possibilities). The first choice is the day for item 1, the second for item 2 and the third for item 3.



   In exactly 2 of the 14 possible outcomes, the first item is not the first to be delivered, If we assume each of these outcomes is equally likely, this has probability $\frac{1}{7} \approx 0.1429$.

   Another approach: There are $3^3 = 27$ ways the three items could be delivered, including the possibility of getting more than one package per day; take this to be our sample space. We will write elements of the sample space as a triple (a, b, c) where a, b and c are the days items 1, 2 and 3 are delivered, respectively (e.g. (1,2,3) is Feb. 1 for item 1, Feb. 2 for item 2, and Feb. 3 for item 3). Assuming that each item will arrive with equal likelihood on each of its 3 days, and that these are independent events, it follows that

$$P(a, b, c) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$$

   for each outcome (a, b, c) in the sample space.

Let M be the event that that the first item is not the first item is not the first to be delivered. Then

$$N = \{(3,2,3),(3,2,4),(3,2,5).\}$$

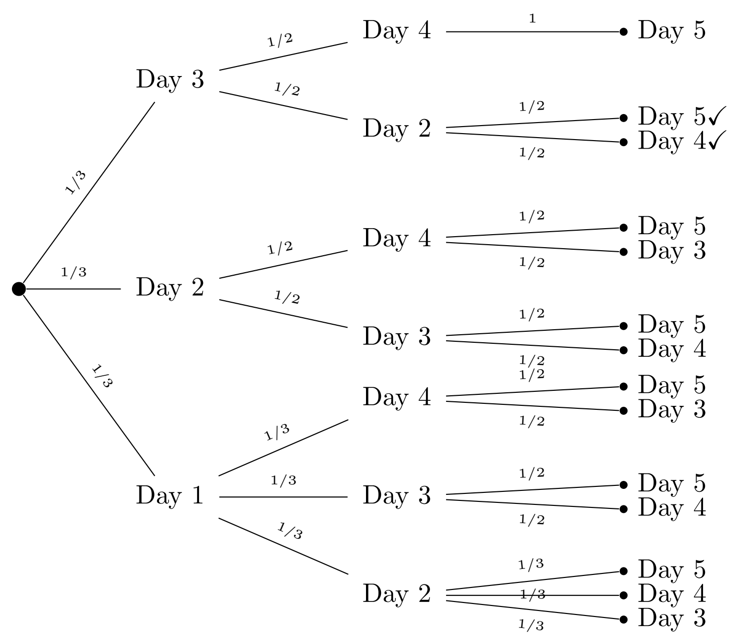
Let M be the event that only one package arrives per day, then

$$M = \{(1,2,3),(1,2,4),(1,2,5),(1,3,4),(1,3,5),(1,4,3),(1,4,5),\\ (2,3,4),(2,3,5),(2,4,3),(2,4,5),(3,2,4),(3,2,5),(3,4,5)\}.$$

Then then probability that the first item is not the first to be delivered, given that at most one item can be delivered in a given day, is

$$P(N|M) = \frac{P(N\cap M)}{P(M)} = \frac{P(\{(3,2,4),(3,2,5)\})}{P(M)} = \frac{\frac{2}{27}}{\frac{14}{27}} = \frac{1}{7}.$$

Another interpretation: If we assume that each package has an equally likely chance of being delivered on each of its 3 days *given* that only one package may be delivered each day, then may compute probabilities using the following tree diagram with probabilities attached to each path (this also assumes the natural order on priority given to item 1, then item 2, then item 3). By the multiplication rule for conditional probability, the probability for any outcome (end point) is the product of the probabilities along the paths taken.



The probabilities for each of the two successful paths are

$$\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot = \frac{1}{12}$$

Therefore the probability that the first item is not delivered first is

$$\frac{1}{12}+\frac{1}{12} = \frac{1}{6} \approx 0.1667.$$

(However, this assignment of probability, while valid, does not yield that each item is delivered on each day with equal likelihood.)

□

6. Let A_1, A_2, A_3, and A_4 be events from a common sample space. These events are pairwise mutually exclusive, with the exception that A_1 and A_2 occur simultaneously with a probability of 0.1. If events A_1, A_2, and A_3 each occur with probability 0.3, what is the largest possible value for the probability of A_4?

   Round to 4 decimal places if needed (do not enter a fraction).

   Answer: 0.2

   Solution. We have given that

$$A_1 \cap A_3 = \emptyset, A_1 \cap A_4 = \emptyset, A_2 \cap A_3 = \emptyset, A_3 \cap A_4 = \emptyset,$$

   and $P(A_1 \cap A_2) = 0.1$. From this we have

$$(A_1 \cup A_2) \cap A_3 = (A_1 \cap A_3) \cup (A_2 \cap A_3) = \emptyset, \quad (A_1 \cup A_2) \cap A_4 = (A_1 \cap A_4) \cup (A_2 \cap A_4) = \emptyset,$$

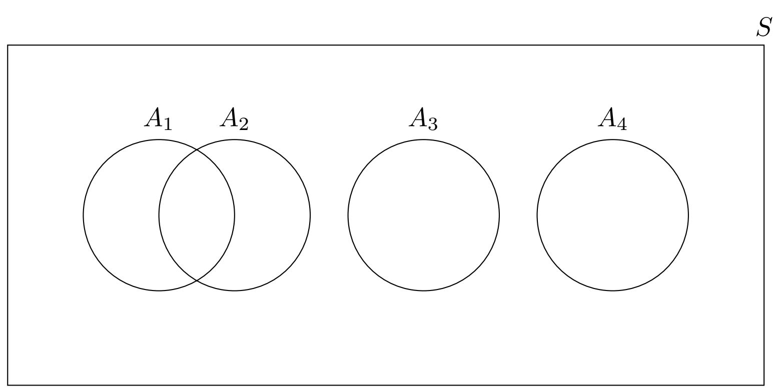
   showing that events $(A_1 \cup A_2)$, A_3, A_4 are mutually exclusive. Now

$$\begin{aligned} 1 &\geq P(A_1 \cup A_2 \cup A_3 \cup A_4) \\ &= P(A_1 \cup A_2) + P(A_3) + P(A_4) \\ &= P(A_1) + P(A_2) - P(A_1 \cap A_2) + P(A_3) + P(A_4) \\ &= (0.3) + (0.3) - (0.1) + (0.3) + P(A_4), \end{aligned}$$

   which implies $1 \geq 0.8 + P(A_4)$, or

$$P(A_4) \leq 0.2.$$

   Here is a Venn diagram describing this setup:



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