MATH1550, Winter 2023
Mini-Assignment 1 – Counting, counting...

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1. How many different "words" can be made using only the letters in the word *UNUSUALLY*.

   Solution. There are 9 letters in total with 3 Us and 2 Ls, and no other repetitions. Therefore the number of different words is
$$\frac{9!}{3! \cdot 2!} = 30240.$$
$\square$

2. A normal coin is tossed ten times and the result, heads or tails, is recorded for each toss. How many possible outcomes are there? (Here we mean to count the different orderings on the 10 flips, and not simply the number of heads and tails that appear.)

   Solution. Viewing this as a 10 step compound event, with 2 choices (heads or tails) at each step, the total number of possible outcomes is
$$2^{10} = 1024.$$
$\square$

3. A normal coin is tossed ten times and the result, heads or tails, is recorded for each toss. How many of the possible outcomes have three heads and seven tails? (Here we mean to count the different orderings on the 10 flips, and not simply the number of heads and tails that appear.)

   Solution. Out of the 10 flips choose 3 for which heads will appear, leaving the rest to be tails. This occurs in
$$\binom{10}{3} = 120$$
ways.

   Equivalently, out of the 10 flips we can choose 7 for which tails will appear. This occurs in
$$\binom{10}{7} = \binom{10}{10-7} = \binom{10}{3} = 120$$
ways. $\square$

4. A normal coin is tossed ten times and the result, heads or tails, is recorded for each toss. How many of the possible outcomes have at least three heads? (Here we mean to count the different orderings on the 10 flips, and not simply the number of heads and tails that appear.)

   Solution. At least 3 heads includes the cases when exactly 3, 4, 5, 6, 7, 8, 9, or 10 heads appear in the 10 flips. This can be counted directly as
$$\binom{10}{3} + \binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}.$$
   Plug this into your calculator, or use the $n = 10$ row of Pascal's Triangle, and we get
$$120 + 210 + 252 + 210 + 120 + 45 + 10 + 1 = 968.$$

An simpler calculation comes from noticing that "at least 3" is the opposite (or *complement*) of getting "at most 2" which means getting 0, 1 or 2 heads, and is counted by

$$\binom{10}{0} + \binom{10}{1} + \binom{10}{2} = 1 + 10 + 45 = 56.$$

Therefore the number of remaining cases (at least 3 heads) is the total number of possible outcomes, minus the ones with at most 2 heads. This is

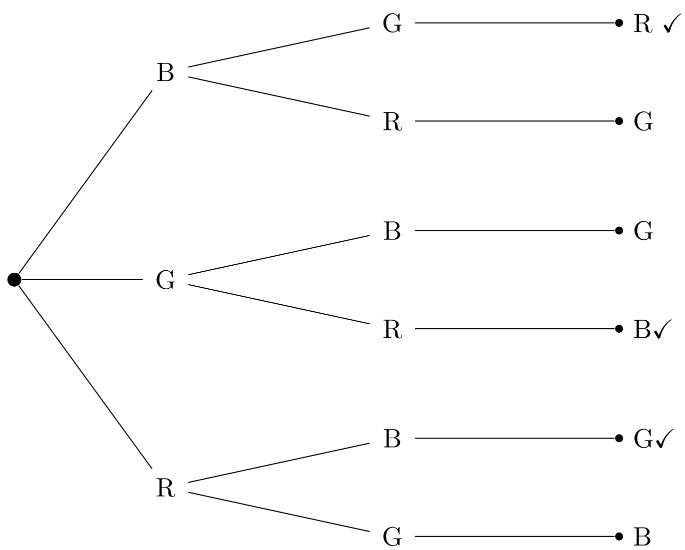
$$2^{10} - 56 = 968.$$

□

5. Toddler triplets, whose names are "Red", "Green", and "Blue", each have a coat whose colour is given by their name, but they aren't clear on which coat belongs to which of them. When they go out, Red chooses a coat at random among the three coats, Green chooses a coat at random among the remaining two coats, and Blue is left to take the remaining coat.

   How many ways are there for exactly one of the toddler triplets to end up with their coat? Hint: you might try drawing a tree diagram to help.

   Solution. In the following tree diagram, Red chooses first, then Blue then Green. We see that there are 3 paths which result in exactly one of the toddlers getting their correct coat.



□

6. We use i to denote an "imaginary" number such that $i^2 = -1$. The *complex numbers* are then the numbers of the form $a + bi$, where a and b are real numbers, and we can add, subtract, and multiply complex numbers as usual, except for that little twist that $i^2 = -1$. Work out what $(2 - i)^6$ is in the form $a + bi$, where a and b are real.

   A: $64 - i$ B: $64 + i$ C: $64 - 64i$ D: $64 + 64i$ E: 0 F: 128

   G: $128i$ H: $-128i$ I: $-117 - 44i$ J: $-117 + 44i$ K: $117 - 44i$ L: Neither

Solution. Recall the binomial theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$

Applying the binomial theorem to $(2-i)^6$ with $n = 6$, $x = 2$ and $y = -i$ we get

$$\begin{aligned}
&(2-i)^6 \\
&= \sum_{r=0}^{6} \binom{6}{r} 2^{6-r}(-i)^r \\
&= \binom{6}{0}2^6 + \binom{6}{1}2^5(-i) + \binom{6}{2}2^4(-i)^2 + \binom{6}{3}2^3(-i)^3 + \binom{6}{4}2^2(-i)^4 + \binom{6}{5}2(-i)^5 + \binom{6}{6}(-i)^6 \\
&= 64 - 6(32)i + 15(16)(-i)^2 + 20(8)(-i)^3 + 15(4)(-i)^4 + 6(2)(-i)^5 + (1)(-i)^6 \\
&= 64 - 192i - 240 + 160i + 60 - 12i - 1 \\
&= -117 - 44i
\end{aligned}$$

To obtain this, we have used the fact that $(-i)^r = (-1)^r(i^r)$, where

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1.$$

□