MATH1550, Winter 2023
Mini-Assignment 11 – Some Common Distributions

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1. Suppose X has a discrete uniform distribution, where the range of X is $\{0, 1, 2, \ldots, 17\}$. Find the variance, $\sigma^2 = \text{var}(X)$, of X.

   Round to 4 decimal places if needed.

   Solution. Note that we have 18 numbers in the range, so the fact that distribution is uniform means that each has a probability of $\frac{1}{18}$ of coming up, *i.e.* $f(x) = P(X = x) = \frac{1}{18}$ for $x \in \{0, 1, 2, \ldots, 17\}$. To compute the variance, we first have to compute the expected value, $E(X)$, of X. We will use the summation formula $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ to minimize the arithmetic a bit.

$$\begin{aligned} E(X) &= \sum_x xf(x) = \sum_{x=0}^{17} x \cdot \frac{1}{18} = \frac{1}{18} \sum_{x=0}^{17} x = \frac{1}{18}(0 + 1 + 2 + \cdots + 17) \\ &= \frac{1}{18} \cdot \frac{17(17+1)}{2} = \frac{1}{18} \cdot \frac{17 \cdot 18}{2} = \frac{17}{2} = 8.5. \end{aligned}$$

   We also need to compute $E\left(X^2\right)$; we will use the summation formula $1^1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ to cut down on the arithmetic required.

$$\begin{aligned} E\left(X^2\right) &= \sum_x x^2 f(x) = \sum_{x=0}^{17} x^2 \cdot \frac{1}{18} = \frac{1}{18} \sum_{x=0}^{17} x^2 = \frac{1}{18}\left(0^2 + 1^2 + 2^2 + \cdots + 17^2\right) \\ &= \frac{1}{18} \cdot \frac{17(17+1)(2 \cdot 17 + 1)}{6} = \frac{1}{18} \cdot \frac{17 \cdot 18 \cdot 35}{6} = \frac{595}{6} \approx 99.1667. \end{aligned}$$

   Finally, we compute the variance:

$$\text{var}(X) = E\left(X^2\right) - [E(X)]^2 = \frac{595}{6} - \left[\frac{17}{2}\right]^2 = \frac{595}{6} - \frac{289}{4} = \frac{1190}{12} - \frac{867}{12} = \frac{323}{12} \approx 26.9167.$$

   □

2. Suppose X has a continuous uniform distribution on the interval $[-1, 3]$. Find the variance, $\sigma^2 = \text{var}(X)$, of X.

   Round to 4 decimal places if needed.

   Solution. Note that the density function of a continuous uniform distribution on the interval $[-1, 3]$ is

$$f(x) = \begin{cases} \frac{1}{4} & -1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases},$$

   since the width of the interval is $3 - (-1) = 4$.

   We need to compute the expected value, $E(X)$, of X first.

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{-1} x \cdot 0\,dx + \int_{-1}^{3} x \cdot \frac{1}{4}\,dx + \int_{3}^{\infty} x \cdot 0\,dx \\
&= 0 + \frac{1}{4}\int_{-1}^{3} x\,dx + 0 = \frac{1}{4} \cdot \frac{x^2}{2}\bigg|_{-1}^{3} = \frac{3^2}{8} - \frac{(-1)^2}{8} = \frac{9}{8} - \frac{1}{8} = 1.
\end{aligned}
$$

After which, we compute $E\left(X^2\right)$.

$$
\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{-1} x^2 \cdot 0\,dx + \int_{-1}^{3} x^2 \cdot \frac{1}{4}\,dx + \int_{3}^{\infty} x^2 \cdot 0\,dx \\
&= 0 + \frac{1}{4}\int_{-1}^{3} x^2\,dx + 0 = \frac{1}{4} \cdot \frac{x^3}{3}\bigg|_{-1}^{3} = \frac{3^3}{12} - \frac{(-1)^3}{12} = \frac{27}{12} - \frac{-1}{12} = \frac{28}{12} = \frac{7}{3} \approx 2.3333.
\end{aligned}
$$

Finally, we compute the variance:

$$
\operatorname{var}(X) = E\left(X^2\right) - [E(X)]^2 = \frac{7}{3} - 1^2 = \frac{4}{3} \approx 1.3333.
$$

Alternative solution: We might have seen in class that a continuous uniform distribution on the interval $[a,b]$ has expected value $E(X) = \frac{a+b}{2}$ and variance $\operatorname{var}(X) = \frac{(b-a)^2}{12}$. In this case, $a = -1$ and $b = 3$, so $E(X) = \frac{-1+3}{2} = \frac{2}{2} = 1$ and $\operatorname{var}(X) = \frac{(3-(-1))^2}{12} = \frac{4^2}{12} = \frac{16}{12} = \frac{4}{3} \approx 1.3333$. □

3. Basketball player Muse will get a ball through the hoop from five meters away 60% of the time. What is the probability that Muse will succeed in getting the ball through the hoop at least three times in five attempts from five meters away?

   Round to 4 decimal places if needed.

   Solution. In this case, the probability of success on each, presumably independent, attempt is $p = \frac{60}{100} = \frac{3}{5} = 0.6$, so the probability of failure on each throw is $q = 1 - p = \frac{2}{5} = 0.4$. Since five throws are taken, we have a binomial distribution with probability distribution function $f(x) = b(x; 5, 0.6) = \binom{5}{x}\left(\frac{3}{5}\right)^x\left(\frac{2}{5}\right)^{5-x}$, where $x \in \{\,0,\,1,\,2,\,3,\,4,\,5\,\}$. It follows that the probability that Muse will succeed in getting the ball through the hoop at least three times in five attempts is

$$
\begin{aligned}
P(\geq 3 \text{ successes}) &= f(3) + f(4) + f(5) \\
&= \binom{5}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^{5-3} + \binom{5}{4}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^{5-4} + \binom{5}{5}\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^{5-5} \\
&= 10 \cdot \frac{27}{125} \cdot \frac{4}{25} + 5 \cdot \frac{81}{625} \cdot \frac{2}{5} + 1 \cdot \frac{243}{3125} \cdot 1 = \frac{1080}{3125} + \frac{810}{3125} + \frac{243}{3125} \\
&= \frac{1080 + 810 + 243}{3125} = \frac{2133}{3125} \approx 0.6826.
\end{aligned}
$$

□

4. Basketball player Muse will get a ball through the hoop from five meters away 60% of the time. What is the minimum number of attempts from five meters away that Muse should make so that the chance of getting at least three balls through the hoop is at least 50%?

Round to 4 decimal places if needed.

Solution. The setup is the same as in question **3**, so we know that with five throws, the probability of Muse getting three balls through the hoop is a bit over 68%. Do four throws suffice to get the probability of Muse putting at least three balls through the hoop over 50%? We basically repeat the calculation in the solution to question **3** for four throws instead of five, so the probability distribution function this time is $f(x) = b(x; 4, 0.6) = \binom{4}{x}\left(\frac{3}{5}\right)^x\left(\frac{2}{5}\right)^{4-x}$, where $x \in \{0, 1, 2, 3, 4\}$. It follows that the probability that Muse will succeed in getting the ball through the hoop at least three times in four attempts is

$$\begin{aligned} P(\geq 3 \text{ successes}) &= f(3) + f(4) \\ &= \binom{4}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^{4-3} + \binom{4}{4}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^{4-4} \\ &= 4 \cdot \frac{27}{125} \cdot \frac{2}{5} + 1 \cdot \frac{81}{625} \cdot 1 = \frac{216}{625} + \frac{81}{625} \\ &= \frac{297}{625} = 0.4752. \end{aligned}$$

With four attempts Muse has a probability less than 0.5 (or 50%) of getting three balls through the hoop, but has a probability of over 68% of getting three balls through the hoop in five attempts. Thus Muse should make at least five attempts to have a probability of over 50% of getting at least three balls through the hoop. □

5. A multiple choice test has ten questions which each have five choices for the answer, with the unusual feature that exactly two of each five are correct. A student who hasn't studied takes this test and picks an answer at random, with equal probability, for each question. If each question is worth 1 point, how many points can this student expect to get on the test?

   Round to 4 decimal places if needed.

   Solution. The student's attempt on each question can be considered as Bernoulli trial with a probability of success, *i.e.* guessing a correct answer, of $p = \frac{2}{5} = 0.4$. There are 10 of these presumably independent trials and the result on the test counts the number of successes, since each correct answer is worth one point. We are thus dealing with a binomial distribution with $n = 10$ trials and a probability of success on each trial of $p = \frac{2}{5} = 0.4$. The expected value of a random variable with this distribution is $E(X) = np = 10 \cdot \frac{2}{5} = 4$, *i.e.* the expected value of this student's result on the test is 4. □

6. A fair standard die is rolled repeatedly until a 6 comes up twice (not necessarily in a row). The random variable X counts the number of rolls made in this process. Find the expected value of X.

   Round to 4 decimal places if needed.

   Solution. Each roll can be thought of as an independent Bernoulli trial with probability of success $p = P(\text{success}) = P(\text{roll } 6) = \frac{1}{6}$ and probability of failure $q = 1 - \frac{1}{6} = \frac{5}{6}$. If X counts the number of trials required until (and including) the second success, then X has a negative binomial distribution. It thus has probability distribution function

$$f(x) = P(2\text{nd success on } x\text{th trial}) = \binom{x-1}{2-1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{x-2} = \binom{x-1}{1}\frac{5^{x-2}}{6^x} = \frac{(x-1)5^{x-2}}{6^x}.$$

Since a negative binomial distribution with probability of success p on each trial and needing k successes has expected value $\mu = \frac{k}{p}$, the expected value of our X is $E(X) = \frac{2}{1/6} = 12$. $\square$