MATH1550, Winter 2023
Mini-Assignment 10 – Moments and Covariance

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1. Let X be a random variable with moment generating function $M_X(t) = (t-1)^{-2}$. Find the variance of X.

   Round to 4 decimal places if needed.

   Solution. We use the moment generating function to compute $E(X)$ and $E\left(X^2\right)$.

$$
\begin{aligned}
E(X) &= \frac{d}{dt}M_X(t)\bigg|_{t=0} = \frac{d}{dt}(t-1)^{-2}\bigg|_{t=0} = -2(t-1)^{-3}\big|_{t=0} = -2(0-1)^{-3} = 2 \\
E\left(X^2\right) &= \frac{d^2}{dt^2}M_X(t)\bigg|_{t=0} = \frac{d^2}{dt^2}(t-1)^{-2}\bigg|_{t=0} = \frac{d}{dt}(-2)(t-1)^{-3}\bigg|_{t=0} \\
&= (-2)(-3)(t-1)^{-4}\big|_{t=0} = 6
\end{aligned}
$$

   Thus $\mu = E(X) = 2$ and $\sigma^2 = \text{var}(X) = E\left(X^2\right) - [E(X)]^2 = 6 - 2^2 = 6 - 4 = 2$. □

2. Suppose that the continuous random variables X and Y have the joint density function

$$
f(x,y) = \begin{cases} 6xy^2 & x,\ y \in [0,1] \\ 0 & \text{otherwise} \end{cases}.
$$

   Find the expected value of X.

   Round to 4 decimal places if needed.

   Solution. Here we go:

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dxdy = \int_0^1\int_0^1 x\cdot 6xy^2\,dxdy + \int\int_{\text{elsewhere}} x\cdot 0\,dxdy \\
&= \int_0^1\int_0^1 6x^2y^2\,dxdy + 0 = \int_0^1 \frac{x^3}{3}\bigg|_0^1 \cdot 6y^2\,dy = \int_0^1 \left(\frac{1^3}{3} - \frac{0^3}{3}\right)6y^2\,dy = \int_0^1 \frac{6}{3}y^2\,dy \\
&= 2\cdot\frac{y^3}{3}\bigg|_0^1 = 2\left(\frac{1^3}{3} - \frac{0^3}{3}\right) = 2\cdot\frac{1}{3} = \frac{2}{3} \approx 0.6667
\end{aligned}
$$

   □

3. Suppose that the continuous random variables X and Y have the joint density function

$$
f(x,y) = \begin{cases} 4xy & x,\ y \in [0,1] \\ 0 & \text{otherwise} \end{cases},
$$

   and suppose the expected values of Y is $E(Y) = \frac{2}{3}$. Find the variance of Y.

Round to 4 decimal places if needed.

Solution. Here we go:

$$\begin{aligned}
E\left(Y^2\right) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x,y)\,dx dy = \int_0^1\int_0^1 y^2\cdot 4xy\,dx dy + \int\int_{\text{elsewhere}} y^2\cdot 0\,dx dy \\
&= \int_0^1\int_0^1 4xy^3\,dx dy + 0 = \int_0^1 \left.\frac{x^2}{2}\right|_0^1 \cdot 4y^3\,dy = \int_0^1 \left(\frac{1^2}{2}-\frac{0^2}{2}\right)4y^3\,dy = \int_0^1 2y^3\,dy \\
&= 2\left.\frac{y^4}{4}\right|_0^1 = 2\left(\frac{1^4}{4}-\frac{0^4}{4}\right) = 2\cdot\frac{1}{4} = \frac{1}{2}
\end{aligned}$$

It follows that $\text{var}(Y) = E\left(Y^2\right) - [E(Y)]^2 = \frac{1}{2} - \left[\frac{2}{3}\right]^2 = \frac{1}{2} - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \frac{1}{18} \approx 0.0556$.
Note that $E\left(Y\right) = E\left(X\right) = \frac{2}{3}$ and $E\left(Y^2\right) = E\left(X^2\right) = \frac{1}{2}$ in this case. (Why? It's because the density function is completely symmetric in x and y, *i.e.* $f(x,y) = f(y,x)$ for all x and y.) ☐

4. Suppose the continuous random variables X and Y have the joint density function

$$f(x,y) = \begin{cases} 4xy & x,\ y\in[0,1] \\ 0 & \text{otherwise} \end{cases}.$$

Suppose the expected values of X and Y are $E(X) = E(Y) = \frac{2}{3}$. Find the covariance of X and Y.

Round to 4 decimal places if needed.

Solution. Here we go:

$$\begin{aligned}
E\left(XY\right) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx dy = \int_0^1\int_0^1 xy\cdot 4xy\,dx dy + \int\int_{\text{elsewhere}} xy\cdot 0\,dx dy \\
&= \int_0^1\int_0^1 4x^2y^2\,dx dy + 0 = \int_0^1 \left.\frac{x^3}{3}\right|_0^1 \cdot 4y^2\,dy = \int_0^1 \left(\frac{1^3}{3}-\frac{0^3}{3}\right)4y^2\,dy = \int_0^1 \frac{4}{3}y^2\,dy \\
&= \left.\frac{4}{3}\frac{y^3}{3}\right|_0^1 = \frac{4}{3}\left(\frac{1^3}{3}-\frac{0^3}{3}\right) = \frac{4}{3}\cdot\frac{1}{3} = \frac{4}{9}
\end{aligned}$$

It follows that $\text{cov}(X,Y) = E\left(XY\right) - E(X)\cdot E(Y) = \frac{4}{9} - \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9} - \frac{4}{9} = 0$. (As it turns out, X and Y are independent, so this result is - dare we say it?! – to be expected.) ☐

5. The joint distribution function $f(x,y)$ of the discrete random variables X and Y is given by the following table.

y \ x	0	1	2	2
1	$\frac{1}{12}$		$\frac{1}{12}$	$\frac{2}{12}$
2		$\frac{1}{12}$	$\frac{1}{12}$	$\frac{2}{12}$
3	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

Find the covariance of X and Y.

Round to 4 decimal places if needed.

Solution. We need to compute $E(X)$, $E(Y)$, and $E(XY)$. Here goes, $E(X)$ first:

$$\begin{aligned}
E(X) &= \sum_x \sum_y x f(x,y) = \sum_x x \left(\sum_y f(x,y) \right) \\
&= 0\left(\frac{1}{12} + 0 + \frac{1}{12}\right) + 1\left(0 + \frac{1}{12} + \frac{1}{12}\right) + 2\left(\frac{1}{12} + \frac{1}{12} + \frac{1}{12}\right) + 3\left(\frac{2}{12} + \frac{2}{12} + \frac{1}{12}\right) \\
&= 0 + \frac{2}{12} + \frac{6}{12} + \frac{15}{12} = \frac{23}{12}
\end{aligned}$$

$E(Y)$ second:

$$\begin{aligned}
E(Y) &= \sum_x \sum_y y f(x,y) = \sum_y \sum_x y f(x,y) = \sum_y y \left(\sum_x f(x,y) \right) \\
&= 1\left(\frac{1}{12} + 0 + \frac{1}{12} + \frac{2}{12}\right) + 2\left(0 + \frac{1}{12} + \frac{1}{12} + \frac{2}{12}\right) + 3\left(\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}\right) \\
&= \frac{4}{12} + \frac{8}{12} + \frac{12}{12} = \frac{24}{12} = 2
\end{aligned}$$

$E(XY)$ third:

$$\begin{aligned}
E(XY) &= \sum_x \sum_y xy f(x,y) = \sum_x x \left(\sum_y y f(x,y) \right) \\
&= 0\left(1 \cdot \frac{1}{12} + 2 \cdot 0 + 3 \cdot \frac{1}{12}\right) + 1\left(1 \cdot 0 + 2 \cdot \frac{1}{12} + 3 \cdot \frac{1}{12}\right) \\
&\quad + 2\left(1 \cdot \frac{1}{12} + 2 \cdot \frac{1}{12} + 3 \cdot \frac{1}{12}\right) + 3\left(1 \cdot \frac{2}{12} + 2 \cdot \frac{2}{12} + 3 \cdot \frac{1}{12}\right) \\
&= 0 \cdot \frac{4}{12} + 1 \cdot \frac{5}{12} + 2 \cdot \frac{6}{12} + 3 \cdot \frac{9}{12} = \frac{44}{12}
\end{aligned}$$

It now follows that $\operatorname{cov}(X,Y) = E(XY) - E(X) \cdot E(Y) = \frac{38}{12} - \frac{23}{12} \cdot 2 = \frac{38-46}{12} = -\frac{8}{12} = -\frac{1}{6} \approx -0.1667$. □

6. Suppose we know that X and Y are random variables with $\operatorname{var}(X) = 2$, $\operatorname{var}(Y) = 3$, and $\operatorname{cov}(X,Y) = -1$. Find $\operatorname{var}(2X - Y)$.

   Round to 4 decimal places if needed.

   Solution. We apply the definition of variance and use the properties of expected value.

$$\begin{aligned}
\operatorname{var}(2X - Y) &= E\left((2X - Y)^2\right) - [E(2X - Y)]^2 = E\left(4X^2 - 4XY + Y^2\right) - [2E(X) - E(Y)]^2 \\
&= 4E\left(X^2\right) - 4E(XY) + E\left(Y^2\right) - \left(4\,[E(X)]^2 - 4E(X)E(Y) + [E(Y)]^2\right) \\
&= 4\left(E\left(X^2\right) - [E(X)]^2\right) - 4\left(E(XY) - E(X)E(Y)\right) + \left(E\left(Y^2\right) - [E(Y)]^2\right) \\
&= 4\operatorname{var}(X) - 4\operatorname{cov}(X,Y) + \operatorname{var}(Y) = 4 \cdot 2 - 4 \cdot (-1) + 3 = 15
\end{aligned}$$

Alternative: We apply the formulas $\mathrm{var}(U+W) = \mathrm{var}(U) + \mathrm{var}(W) + 2\,\mathrm{cov}(U,W)$, $\mathrm{cov}(aU,bW) = ab\,\mathrm{cov}(U,W)$, and $\mathrm{var}(aU) = a^2\mathrm{var}(U)$ for any constant a.

$$\begin{aligned}\mathrm{var}\,(2X - Y) &= \mathrm{var}\,(2X + (-1)Y) = \mathrm{var}\,(2X) + \mathrm{var}\,((-1)Y) + 2\,\mathrm{cov}\,(2X, (-1)Y) \\ &= 2^2\mathrm{var}(X) + (-1)^2\mathrm{var}(Y) + 2\cdot 2\cdot(-1)\,\mathrm{cov}(X,Y) \\ &= 4\mathrm{var}(X) + \mathrm{var}(Y) - 4\mathrm{cov}(X,Y) = 4\cdot 2 + 3 - 4\cdot(-1) = 15\end{aligned}$$

□