MATH1550, Winter 2023:
Exercise Set 9

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1. Let X be a random variable with the following distribution

x	-3	-1	2	5
$P(X=x)$	0.3	0.1	0.2	0.4

   (a) Find the expected value of X.
   (b) Find the variance of X.
   (c) Find the 3rd moment about the mean of X.

2. Let Y be a random variable with the following distribution

y	2	3	4	5	6
$P(Y=y)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

   (a) Find the expected value of Y.
   (b) Find the variance of Y.
   (c) Find the 3rd moment about the mean of Y.

3. Let X be a continuous random variable with probability density

$$f(x) = \begin{cases} \frac{1}{10}(3x^2+1) & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the mean and variance of X.

4. Write the definition for the 3rd moment about the mean, and then devise a "shortcut" formula in terms of the moments about the origin. Do this using properties of expected value as was done to obtain a formula for the second moment about the mean.

5. Derive an expression for $E((X-\mu)^4))$ which involves only terms $E(X^4), E(X^3), E(X^2), E(X)$. In other words, find a "shortcut" formula which allows us to compute $E((X-\mu)^4))$ from moments around the origin.

6. Find $\mu = E(X)$, $E(X^2)$, σ^2 (variance) and σ (standard deviation) for the discrete random variable X that has the probability distribution $f(x) = \frac{1}{2}$ for $x = -2$ and $x = 2$.

7. If the probability density of X is given by

$$f(x) = \begin{cases} 630x^4(1-x)^4 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   find the probability that X will take on a value within two standard deviations of the mean and compare this probability with the lower bound provided by Chebyshev's Theorem.

8. A study of the nutritional value of a certain kind of bread shows that the amount of thiamine (vitamin B_1) in a slice may be looked upon as a random variable X with $\mu = 0.260$ milligrams and $\sigma = 0.005$ milligrams. According to Chebyshev's Theorem, what interval of thiamine content values about μ must we consider, in order to include:

   (a) at least 35 of every 36 slices of bread?

   (b) at least 143 of every 144 slices of bread?

9. Let X be a continuous random variable with probability density

$$f(x) = \begin{cases} \frac{1}{6}x + \frac{1}{12} & \text{for } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the mean μ of X.

   (b) Find the variance σ^2 of X.

   (c) Compute $P(1 \leq X \leq 2)$.

   (d) Find $P(|X - \mu| < \frac{3}{2}\sigma)$, and compare this value with what Chebyshev's Theorem tells us.

10. Let X be a continuous random variable with probability density given by

$$f(x) = \begin{cases} \frac{1}{8}(x + 1) & \text{for } 2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the mean μ of X.

   (b) Find the variance of X.

   (c) Find the 3rd moment about the mean for X.

   (d) Find the standard deviation σ, and find $P(|X - \mu| < 2\sigma)$.

11. Let X be a discrete random variable with the probability distribution given below. Find the variance of X.

x	$f(x)$
-2	$\frac{1}{20}$
-1	$\frac{3}{20}$
0	$\frac{6}{20}$
1	$\frac{2}{20}$
2	$\frac{7}{20}$
3	$\frac{1}{20}$

12. Let X be a discrete random variable with the probability distribution given below. Find the third moment about the mean of X.

x	$f(x)$
-2	$\frac{1}{20}$
-1	$\frac{3}{20}$
0	$\frac{6}{20}$
1	$\frac{2}{20}$
2	$\frac{7}{20}$
3	$\frac{1}{20}$

13. Let X be a continuous random variable with the probability density given below. Compute the variance of X.

$$f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

14. A random variable X has mean $\mu = 124$ and standard deviation $\sigma = 7.5$. According to Chebyshev's Theorem, what is the minimum probability that X lies between 64 and 184?

15. Let X be a discrete random variable with the probability distribution given below. What does Chebyshev's Theorem tell us is the minimum probability that X lies within 1.3 standard deviations of the mean?

x	$f(x)$
-2	$\frac{1}{20}$
-1	$\frac{3}{20}$
0	$\frac{6}{20}$
1	$\frac{2}{20}$
2	$\frac{7}{20}$
3	$\frac{1}{20}$

16. Let X be a continuous random variable with probability density

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the moment generating function $M_X(t)$ of X.

17. Prove the following properties of moment generating functions:

(a)
$$M_{X+a}(t) = e^{at} \cdot M_X(t)$$

(b)
$$M_{bX}(t) = M_X(bt)$$

(c)
$$M_{\frac{X+a}{b}}(t) = e^{\frac{a}{b}t} \cdot M_X\left(\frac{t}{b}\right)$$

18. Let X be a continuous random variable with probability density given by

$$f(x) = \begin{cases} 3e^{-3x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the moment generating function for X. (Hint: Express the integrand as $e^{-x(t-3)}$ and restrict $t < 3$.)

(b) Use the moment generating function to find the mean and variance of X.

19. Suppose the continuous random variable X has moment generating function given by

$$M_X(t) = 2(2-t)^{-1}$$

for $-2 < t < 2$. Find the mean and variance of X.

20. Find the moment generating function of the discrete random variable X that has probability distribution

$$f(x) = 2\left(\frac{1}{3}\right)^x, \quad \text{for } x \in \mathbb{N},$$

and use it to find the mean an variance of X. *Hint: Use the formula for sum of an infinite geometric series* $\sum_{i=0}^{\infty} ar^i = \frac{a}{a-r}$

21. Suppose a random variable X has moment generating function

$$M_X(t) = e^{3t+8t^2}.$$

Find the mean and the variance of X.

22. Let X be a random variable with moment generating function

$$M_X(t) = \frac{1}{1-t^2}.$$

Find the mean of X.

23. Let X be a random variable with moment generating function

$$M_X(t) = \frac{1}{1-t^2}.$$

Find the variance of X.