

**MATH1550, Winter 2023:**  
**Exercise Set 8**

---

1. Let  $X$  be a discrete random variable with the following probability distribution:

$$P(X = 0) = \frac{1}{3}, \quad P(X = 1) = P(X = 6) = \frac{1}{165},$$

$$P(X = 2) = P(X = 5) = \frac{1}{11}, \quad P(X = 3) = P(X = 4) = \frac{13}{55}$$

Find  $E(X)$ .

*Solution.*

$$E(x) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{165} + 2 \cdot \frac{1}{11} + 3 \cdot \frac{13}{55} + 4 \cdot \frac{13}{55} + 5 \cdot \frac{1}{11} + 6 \cdot \frac{1}{165} = \frac{7}{3}$$

□

2. Two coins are tossed. The first coin has a probability of 0.6 that it will land on heads, and the second coin has a probability of 0.7 that it will land on heads. Let  $X$  be the total number of heads.

- (a) What is the range of  $X$ ?  
(b) Find the probability distribution for  $X$ .  
(c) Compute  $E(X)$ .

*Solution.* (a) The range of  $X$  is:

$$\{0, 1, 2\}$$

(b)

$$P(X = 0) = (0.4)(0.3) = 0.12$$

$$P(X = 1) = (0.6)(0.3) + (0.4)(0.7) = 0.46$$

$$P(X = 2) = (0.6)(0.7) = 0.42$$

or

$x$	$P(X = x)$
0	0.12
1	0.46
2	0.42

(c)

$$E(X) = 0 \cdot (0.12) + 1 \cdot (0.46) + 2 \cdot (0.42) = 1.3$$

□

3. You are playing a dice game where two (regular) dice are rolled and you are paid the amount shown (the sum of the two dice) in dollars. If the game costs \$7 to play, what can you expect to win or lose; i.e. what is the expected value of this game?

*Solution.* Let  $X$  be the sum of both dice. The range of  $X$  is  $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$  and as we have seen in earlier examples the probability distribution for  $X$  is given by

$$f(x) = \frac{6 - |7 - x|}{36}.$$

The expected value of  $X$  is

$$\begin{aligned} E(X) &= \sum_{x=2}^{12} x \cdot f(x) \\ &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + 9 \cdot \frac{4}{36} + 10 \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} \\ &= \frac{252}{36} \\ &= 7. \end{aligned}$$

Since it costs \$7 to play the game and our expected payout is \$7, we can expect to break even in the long run.

We can arrive at this in a different way. Let  $Y$  be the profit made from each roll; that is  $Y$  is the sum of both dice minus 7. Then the range of  $Y$  is  $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ . The probability distribution for  $Y$  is  $g(y) = P(Y = y) = P(X = y + 7) = f(y + 7) = \frac{6 - |-y|}{36}$ . The expected value of  $Y$  is

$$\begin{aligned} E(Y) &= \sum_{y=-5}^5 y \cdot g(y) \\ &= (-5) \cdot \frac{1}{36} + (-4) \cdot \frac{2}{36} + (-3) \cdot \frac{3}{36} + (-2) \cdot \frac{4}{36} + (-1) \cdot \frac{5}{36} + 0 \cdot \frac{6}{36} \\ &\quad + 1 \cdot \frac{5}{36} + 2 \cdot \frac{4}{36} + 3 \cdot \frac{3}{36} + 4 \cdot \frac{2}{36} + 5 \cdot \frac{1}{36} \\ &= 0. \end{aligned}$$

Therefore our expected profit is 0. We might have seen this coming because of the symmetry in both the values for  $Y$ , and its probability distribution.  $\square$

4. You run a business buying and selling coconuts. You have \$1,000, and coconuts are currently selling for \$2 each. In one week you can sell the coconuts, but the price will change to either half the price (\$1) or double the price (\$4), with each of these being equally likely.
- If your goal is maximize the *expected* amount of money you have after a week (i.e. after you are able to sell) how many coconuts should you buy at \$2 each?
  - If your goal is maximize your *expected* number of coconuts after a week, how many coconuts should you buy at \$2 each (vs. buying a week later at the new price)?

*Solution.* (a) Let  $n$  be the number of coconuts that you buy at \$2 apiece, and let  $X$  be the random variable whose value is the total amount of money you have after a week. Then the range of  $X$  is

$$\{(1000 - 2n) + n, (1000 - 2n) + 4n\} = \{1000 - n, 1000 + 2n\}.$$

Each outcome is assumed equally likely, so

$$P(X = (1000 - n)) = 0.5, \quad P(X = (1000 + 2n)) = 0.5.$$

The expected amount of money after a week is

$$E(X) = (1000 - n)(0.5) + (1000 + 2n)(0.5) = 1000 + (0.5)n.$$

This shows that the more coconuts we buy at \$2 (the larger  $n$  is) the more money we can expect to have after a week. Therefore we should buy 500 coconuts (by spending all of the \$1000 we initially had).

- (b) Let  $Y$  be the total number of coconuts we have after a week (i.e. the number we buy at \$2 each, plus the number we buy at the new price). The range of  $Y$  is

$$\left\{ n + (1000 - 2n), n + \frac{1}{4}(1000 - 2n) \right\} = \left\{ 1000 - n, 250 + \frac{n}{2} \right\}$$

Again, each of these outcomes is equally likely. The expected total number of coconuts is

$$E(Y) = (1000 - n)(0.5) + \left( 250 + \frac{n}{2} \right) (0.5) = 625 - \frac{1}{4}n.$$

This shows that buying any positive number of coconuts at \$2 each decreases the expected total number after a week. Therefore we shouldn't buy any coconuts at \$2 each, and wait to buy them after a week.

□

5. A game of chance is called *fair*, if each player's expected value is zero. If the casino pays us \$10 for rolling a 3 or a 4 with a regular 6-sided die, what should we have to pay for rolling a 1,2,5, or 6, in order to make this a fair game?

*Solution.* Let  $X$  be the random variable which gives the payout for each roll of the game. If we assume that the payout is  $k$  for each of 1,2,5, or 6 (i.e. it is the same amount for each) then  $X$  has range  $\{k, 10\}$ . The probability distribution for  $X$  is

$$f(x) = \begin{cases} \frac{2}{6} = P(\{3, 4\}) & \text{for } x = 10 \\ \frac{4}{6} = P(\{1, 2, 5, 6\}) & \text{for } x = k. \end{cases}$$

So the expected value for  $X$  is,

$$\begin{aligned} E(X) &= k \cdot f(k) + 10 \cdot f(10) \\ &= k \cdot \frac{4}{6} + 10 \cdot \frac{2}{6} \\ &= \frac{2}{3}(k + 5). \end{aligned}$$

For this to be a fair game we must have  $0 = E(X) = \frac{2}{3}(k + 5)$ . This implies that  $k = -5$ ; i.e. we must pay \$5 any time we roll a 1,2,5, or 6. □

6. The probability density of  $X$  is given by

$$f(x) = \begin{cases} \frac{1}{8}(x + 1) & \text{for } 2 \leq x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

Find the mean  $\mu$  of  $X$ . (Note that  $\mu = E(X)$ .)

*Solution.*

$$\begin{aligned}\mu &= E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx \\ &= \int_2^4 x \cdot \frac{1}{8}(x+1) dx \\ &= \frac{1}{8} \int_2^4 x^2 + x dx \\ &= \frac{1}{8} \left( \frac{x^3}{3} + \frac{x^2}{2} \right) \Big|_2^4 \\ &= \frac{1}{8} \left( \frac{64}{3} + \frac{16}{2} - \frac{8}{3} - \frac{4}{2} \right) \\ &= \frac{37}{12}\end{aligned}$$

□

7. Let  $X$  be a random variable with the following distribution

$x$	-2	-1	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Let  $Y = X^2$ .

- (a) Find the distribution  $g(y)$  of  $Y$ .
- (b) Find the joint distribution  $f(x, y)$  of  $X$  and  $Y$ .
- (c) Find the expected value of  $2X + Y$ .
- (d) Find  $E(X)$ ,  $E(Y)$  and  $E(XY)$ . Note that this example shows that  $E(XY) = E(X)E(Y)$ , however  $X$  and  $Y$  are not independent.

*Solution.* (a) Since  $Y = X^2$ , the range of  $Y$  is  $\{1, 4\}$ , and

$$P(Y = 1) = P(X = -1) + P(X = 1), \quad P(Y = 4) = P(X = -2) + P(X = 2).$$

In summary, the distribution for  $Y$  is

$y$	1	4
$P(Y = y)$	$\frac{1}{2}$	$\frac{1}{2}$

- (b) The joint distribution is

		$x$			
		-2	-1	1	2
$y$	1	0	$\frac{1}{4}$	$\frac{1}{4}$	0
	4	$\frac{1}{4}$	0	0	$\frac{1}{4}$

(c)

$$\begin{aligned} E(2X + Y) &= \sum_x \sum_y (2x + y) \cdot f(x, y) \\ &= (2(-2) + 1) \cdot f(-2, 1) + (2(-2) + 4) \cdot f(-2, 4) + (2(-1) + 1) \cdot f(-1, 1) \\ &\quad + (2(-1) + 4) \cdot f(-1, 4) + (2(1) + 1) \cdot f(1, 1) + (2(1) + 4) \cdot f(1, 4) \\ &\quad + (2(2) + 1) \cdot f(2, 1) + (2(2) + 4) \cdot f(2, 4) \\ &= (-3) \cdot 0 + (0) \cdot \frac{1}{4} + (-1) \cdot \frac{1}{4} + (2) \cdot 0 + (3) \cdot \frac{1}{4} + (6) \cdot 0 + (5) \cdot 0 + (8) \cdot \frac{1}{4} \\ &= 2.5. \end{aligned}$$

(d) Let  $g(x)$  denote the probability distribution of  $X$ , and  $h(y)$  denote the probability distribution for  $Y$ . Then

$$E(X) = \sum_x x \cdot h(x) = (-2) \cdot \frac{1}{4} + (-1) \cdot \frac{1}{4} + (1) \cdot \frac{1}{4} + (2) \cdot \frac{1}{4} = 0.$$

$$E(Y) = \sum_y y \cdot g(y) = (1) \cdot \frac{1}{2} + (4) \cdot \frac{1}{2} = 2.5.$$

We also have

$$\begin{aligned} E(XY) &= \sum_x \sum_y (xy) \cdot f(x, y) \\ &= ((-2) \cdot 1) \cdot f(-2, 1) + ((-2) \cdot 4) \cdot f(-2, 4) + ((-1) \cdot 1) \cdot f(-1, 1) \\ &\quad + ((-1) \cdot 4) \cdot f(-1, 4) + (1 \cdot 1) \cdot f(1, 1) + (1 \cdot 4) \cdot f(1, 4) \\ &\quad + (2 \cdot 1) \cdot f(2, 1) + (2 \cdot 4) \cdot f(2, 4) \\ &= (-2) \cdot 0 + (-8) \cdot \frac{1}{4} + (-1) \cdot \frac{1}{4} + (-4) \cdot 0 + (1) \cdot \frac{1}{4} + (4) \cdot 0 + (2) \cdot 0 + (8) \cdot \frac{1}{4} \\ &= 0. \end{aligned}$$

Note that  $f(-2, 1) = 0 \neq g(-2) \cdot h(1) = \frac{1}{4} \cdot \frac{1}{2}$  and therefore  $X$  and  $Y$  are not independent, but clearly  $E(XY) = E(X) \cdot E(Y)$ . We proved in class that independence implies  $E(XY) = E(X) \cdot E(Y)$ , but the reverse implication does not hold, as this example shows.

□

8. The joint probability density of  $X$  and  $Y$  is given by

$$f(x, y) = \begin{cases} \frac{2}{7}(x + 2y) & \text{for } 0 \leq x \leq 1, 1 < y < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the expected value of  $g(X, Y) = \frac{X}{Y^3}$ .

*Solution.*

$$\begin{aligned}
 E\left(\frac{X}{Y^3}\right) &= \int_1^2 \int_0^1 \frac{2x(x+2y)}{7y^3} dx dy \\
 &= \frac{2}{7} \int_1^2 \left. \frac{x^3}{3y^3} + \frac{x^2}{y^2} \right|_0^1 dy \\
 &= \frac{2}{7} \int_1^2 \frac{1}{3y^3} + \frac{1}{y^2} dy \\
 &= \frac{2}{7} \left[ -\frac{1}{6y^2} - \frac{1}{y} \right]_1^2 \\
 &= \frac{5}{28}
 \end{aligned}$$

□

9. The probability density of  $X$  is given by

$$f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x \leq 1 \\ \frac{1}{2} & \text{for } 1 < x \leq 2 \\ \frac{3-x}{2} & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the expected value of  $g(X) = X^2 - 5X + 3$ .

*Solution.*

$$\begin{aligned}
 E(X^2 - 5X + 3) &= \int_{-\infty}^{\infty} (x^2 - 5x + 3)f(x) dx \\
 &= \int_0^1 (x^2 - 5x + 3) \left(\frac{x}{2}\right) dx + \int_1^2 (x^2 - 5x + 3) \left(\frac{1}{2}\right) dx \\
 &\quad + \int_2^3 (x^2 - 5x + 3) \left(\frac{3-x}{2}\right) dx \\
 &= \frac{1}{2} \left[ \frac{x^4}{4} - \frac{5x^3}{3} + \frac{3x^2}{2} \right]_0^1 + \frac{1}{2} \left[ \frac{x^3}{3} - \frac{5x^2}{2} + 3x \right]_1^2 \\
 &\quad + \frac{3}{2} \left[ \frac{x^3}{3} - \frac{5x^2}{2} + 3x \right]_2^3 - \frac{1}{2} \left[ \frac{x^4}{4} - \frac{5x^3}{3} + \frac{3x^2}{2} \right]_2^3 \\
 &= \frac{1}{2} \left[ \frac{1}{4} - \frac{5}{3} + \frac{3}{2} \right] + \frac{1}{2} \left[ \left( \frac{8}{3} - 10 + 6 \right) - \left( \frac{1}{3} - \frac{5}{2} + 3 \right) \right] \\
 &\quad + \frac{3}{2} \left[ \left( 9 - \frac{45}{2} + 9 \right) - \left( \frac{8}{3} - 10 + 6 \right) \right] - \frac{1}{2} \left[ \left( \frac{81}{4} - 45 + \frac{27}{2} \right) - \left( 4 - \frac{40}{3} + 6 \right) \right] \\
 &= \frac{1}{24} - \frac{13}{12} - \frac{19}{4} + \frac{95}{24} \\
 &= -\frac{11}{6}.
 \end{aligned}$$

□

10. The probability that Ms. Brown will sell a piece of property at a profit of \$3,000 is  $\frac{3}{20}$ , the probability that she will sell at a profit of \$1,500 is  $\frac{7}{20}$ , the probability that she will break even is  $\frac{7}{20}$  and the probability that she will lose \$1,500 is  $\frac{3}{20}$ . What is her expected profit?

*Solution.* Let random variable  $X$  represent the profit of the sale. Then range of  $X$  is  $\{-1500, 0, 1500, 3000\}$  and the probability distribution for  $X$  is

$x$	$P(X = x)$
-1500	$\frac{3}{20}$
0	$\frac{7}{20}$
1500	$\frac{7}{20}$
3000	$\frac{3}{20}$

Thus the expected profit is (in dollars)

$$E(X) = (-1500) \cdot \frac{3}{20} + (0) \cdot \frac{7}{20} + (1500) \cdot \frac{7}{20} + (3000) \cdot \frac{3}{20} = 750.$$

□

11. The joint probability density of  $X$  and  $Y$  is given by

$$f(x, y) = \begin{cases} \frac{2}{5}(2x + 3y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find  $E(XY)$ .

*Solution.*

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y) \, dx \, dy \\ &= \int_0^1 \int_0^1 xy \left( \frac{2}{5}(2x + 3y) \right) \, dx \, dy \\ &= \frac{2}{5} \int_0^1 \left[ \frac{2x^3y}{3} + \frac{3}{2}x^2y^2 \right]_0^1 \, dy \\ &= \frac{2}{5} \int_0^1 \frac{2y}{3} + \frac{3}{2}y^2 \, dy \\ &= \frac{2}{5} \left[ \frac{y^2}{3} + \frac{y^3}{2} \right]_0^1 \\ &= \frac{1}{3}. \end{aligned}$$

□

12. The number of minutes that a flight from Phoenix to Tucson is early or late is a continuous random variable with probability density

$$f(x) = \begin{cases} \frac{1}{243}(36 - x^2) & \text{for } -6 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

If the posted arrival time is 12:00pm, find the expected arrival time. (Take negative values to mean early, positive values to mean late)

*Solution.* Let  $X$  be the continuous random variable representing the number of minutes past 12:00 that the flight arrives. Then

$$\begin{aligned}
 E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) dx \\
 &= \int_{-\infty}^{-6} x \cdot f(x) dx + \int_{-6}^3 x \cdot f(x) dx + \int_3^{\infty} x \cdot f(x) dx \\
 &= \int_{-\infty}^{-6} x \cdot 0 dx + \int_{-6}^3 x \cdot \left( \frac{1}{243}(36 - x^2) \right) dx + \int_3^{\infty} x \cdot 0 dx \\
 &= \int_{-6}^3 \frac{36x}{243} - \frac{x^3}{243} dx \\
 &= \left. \frac{18x^2}{243} - \frac{x^4}{972} \right|_{-6}^3 \\
 &= \frac{18(9)}{243} - \frac{81}{972} - \frac{18(36)}{243} + \frac{1296}{972} \\
 &= -0.75
 \end{aligned}$$

Therefore the expected arrival time is 0.75 minutes, or 45 seconds, before 12:00pm.  $\square$

13. Find the expected value for a random variable  $X$  with probability density function given by

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$$

*Solution.*

$$\begin{aligned}
 E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) dx \\
 &= \int_{-\infty}^0 x \cdot f(x) dx + \int_0^1 x \cdot f(x) dx + \int_1^2 x \cdot f(x) dx + \int_2^{\infty} x \cdot f(x) dx \\
 &= \int_{-\infty}^0 x \cdot 0 dx + \int_0^1 x \cdot x dx + \int_1^2 x \cdot (2 - x) dx + \int_2^{\infty} x \cdot 0 dx \\
 &= \int_0^1 x^2 dx + \int_1^2 2x - x^2 dx \\
 &= \left( \frac{x^3}{3} \right) \Big|_0^1 + \left( x^2 - \frac{x^3}{3} \right) \Big|_1^2 \\
 &= \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3} \\
 &= 1.
 \end{aligned}$$

$\square$

14. Let  $X$  be the number of points rolled with a regular 6-sided die. Find the expected value of  $3X^2 + 2X - 1$ .

*Solution.* Let  $g(X) = 3X^2 + 2X - 1$ . Then, from our theorem on expected value of a function of a random variable, we have

$$E(g(X)) = \sum_x g(x) \cdot f(x),$$



where the sum is taken over all values in the range of  $X$ , namely  $\{1, 2, 3, 4, 5, 6\}$ , and  $f(x) = \frac{1}{6}$  (for all  $x$ ) is the probability distribution of  $X$ . Thus

$$\begin{aligned} E(g(X)) &= \sum_{x=1}^6 (3x^2 + 2x - 1) \cdot \frac{1}{6} \\ &= \frac{(3(1)^2 + 2(1) - 1)}{6} + \frac{(3(2)^2 + 2(2) - 1)}{6} + \frac{(3(3)^2 + 2(3) - 1)}{6} \\ &\quad + \frac{(3(4)^2 + 2(4) - 1)}{6} + \frac{(3(5)^2 + 2(5) - 1)}{6} + \frac{(3(6)^2 + 2(6) - 1)}{6} \\ &= \frac{4}{6} + \frac{15}{6} + \frac{32}{6} + \frac{55}{6} + \frac{84}{6} + \frac{119}{6} \\ &= 51.5. \end{aligned}$$

□

15. Let  $X$  be a discrete random variable with the probability distribution given below. Find the expected value of  $X$ .

$x$	$f(x)$
-2	$\frac{1}{20}$
-1	$\frac{3}{20}$
0	$\frac{6}{20}$
1	$\frac{2}{20}$
2	$\frac{7}{20}$
3	$\frac{1}{20}$

*Solution.*

$$E(X) = (-2)\frac{1}{20} + (-1)\frac{3}{20} + (0)\frac{6}{20} + (1)\frac{2}{20} + (2)\frac{7}{20} + (3)\frac{1}{20} = \frac{14}{20}.$$

□

16. Let  $X$  be a discrete random variable with the probability distribution given below. Find  $E(X^2)$ .

$x$	$f(x)$
-2	$\frac{1}{20}$
-1	$\frac{3}{20}$
0	$\frac{6}{20}$
1	$\frac{2}{20}$
2	$\frac{7}{20}$
3	$\frac{1}{20}$

*Solution.*

$$E(X^2) = (-2)^2\frac{1}{20} + (-1)^2\frac{3}{20} + (0)^2\frac{6}{20} + (1)^2\frac{2}{20} + (2)^2\frac{7}{20} + (3)^2\frac{1}{20} = \frac{46}{20}.$$

□

17. Let  $X$  be a continuous random variable with the probability density given below. Find  $E(X)$ .

$$f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 \leq x \leq 1 \\ \frac{1}{2} & \text{for } 1 < x \leq 2 \\ \frac{3-x}{2} & \text{for } 2 < x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

*Solution.*

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x) dx \\ &= \int_0^1 \frac{x^2}{2} dx + \int_1^2 \frac{x}{2} dx + \int_2^3 \frac{3x-x^2}{2} dx \\ &= \left[ \frac{x^3}{6} \right]_0^1 + \left[ \frac{x^2}{4} \right]_1^2 + \left[ \frac{3x^2}{4} - \frac{x^3}{6} \right]_2^3 \\ &= \frac{1}{6} - 0 + 1 - \frac{1}{4} + \frac{27}{4} - \frac{27}{6} - 3 + \frac{8}{3} \\ &= \frac{17}{6} \end{aligned}$$

□

18. A game of chance is called *fair* if each player's expected profit is zero. Consider a casino game where the player rolls two fair dice and wins the sum shown on the 2 dice (in dollars). How much should the casino charge the player in order to make this a fair game?

*Solution.* Let  $X$  be the sum of the two dice, and  $f(x)$  its distribution. Then.

$$\begin{aligned} E(X) &= \sum_{x=2}^{12} xf(x) \\ &= \sum_{x=2}^{12} x \left( \frac{6-|7-x|}{36} \right) \\ &= \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36} \\ &= 7 \end{aligned}$$

Therefore the casino should charge \$7 for one play in order to make this a fair game.

□

19. A game of chance is called *fair* if each player's expected profit is zero. Suppose you are making wagers with your friend and you tell them that they have to pay you \$10 if they roll a 3 or a 4 with fair 6-sided die. In order to make the game fair, how much should your promise to pay your friend if they roll a 1, 2, 5, or 6?

*Solution.* Let  $X$  be the amount of money that you win by playing the game, and  $p$  be the amount you pay to your friend for rolling a 1,2,5 or 6; i.e.  $-p$  is the amount you "win" on those numbers. The range of  $X$  is then  $\{-p, 10\}$  and the expected winnings are

$$E(X) = (-p)\frac{4}{6} + 10\frac{2}{6}.$$

Thus if the game is to be fair, then

$$(-p)\frac{4}{6} + 10\frac{2}{6} = 0 \Rightarrow p = 5.$$

So you must agree to pay \$5 if your friend rolls 1,2,5 or 6. □

20. The joint distribution for  $X$  and  $Y$  is given below. Find  $E(X + Y)$ .

		$x$	
		0	1
$y$	0		$\frac{1}{8}$
	1	$\frac{1}{8}$	$\frac{2}{8}$
	2	$\frac{2}{8}$	$\frac{1}{8}$
	3	$\frac{1}{8}$	

*Solution.*

$$E(X + Y) = \sum_{x=0}^1 \sum_{y=0}^3 (x + y)f(x, y) = (1)\frac{1}{8} + (2)\frac{2}{8} + (3)\frac{1}{8} + (1)\frac{1}{8} + (2)\frac{2}{8} + (3)\frac{1}{8} = 2.$$

□