MATH1550, Winter 2023:
Exercise Set 7

---

1. Let X and Y be discrete random variables with joint probability distribution given by the following table:

y \ x	1	2	3	4
0	$\frac{4}{84}$	$\frac{18}{84}$	$\frac{12}{84}$	$\frac{1}{84}$
1	$\frac{12}{84}$	$\frac{24}{84}$	$\frac{6}{84}$	
2	$\frac{4}{84}$	$\frac{3}{84}$		

(a) Find the marginal distributions for X and Y.
(b) Find the conditional distribution for X given $Y = 1$.

Solution. (a) The marginal distribution $g(x)$ for X is given by

$$\begin{aligned} g(1) &= \frac{4}{84} + \frac{12}{84} + \frac{4}{84} = \frac{20}{84}, \\ g(2) &= \frac{18}{84} + \frac{24}{84} + \frac{3}{84} = \frac{45}{84}, \\ g(3) &= \frac{12}{84} + \frac{6}{84} = \frac{18}{84}, \\ g(4) &= \frac{1}{84}. \end{aligned}$$

The marginal distribution $h(y)$ for Y is given by

$$\begin{aligned} h(0) &= \frac{4}{84} + \frac{18}{84} + \frac{12}{84} + \frac{1}{84} = \frac{35}{84}, \\ h(1) &= \frac{12}{84} + \frac{24}{84} + \frac{6}{84} = \frac{42}{84}, \\ h(2) &= \frac{4}{84} + \frac{3}{84} = \frac{7}{84}. \end{aligned}$$

(b) If we let $f(x, y)$ denote the joint distribution of X and Y, then the conditional distribution for X given $Y = 1$ is defined as,

$$f(x|1) = \frac{f(x,1)}{h(1)} = \frac{f(x,1)}{\left(\frac{42}{84}\right)}.$$

So we have

$$\begin{aligned} f(1|1) &= \frac{\left(\frac{12}{84}\right)}{\left(\frac{42}{84}\right)} = \frac{12}{42}, \\ f(2|1) &= \frac{\left(\frac{24}{84}\right)}{\left(\frac{42}{84}\right)} = \frac{24}{42}, \\ f(3|1) &= \frac{\left(\frac{6}{84}\right)}{\left(\frac{42}{84}\right)} = \frac{6}{42}, \\ f(4|1) &= \frac{0}{\left(\frac{42}{84}\right)} = 0. \end{aligned}$$

□

2. A fair coin is tossed twice. Let X and Y be random variables such that

   - $X = 1$ if the first toss is heads, and $X = 0$ otherwise.
   - $Y = 1$ if both tosses are heads, and $Y = 0$ otherwise

   (a) Give the joint probability distribution for X and Y
   (b) Find the marginal distributions for X and Y.
   (c) Determine whether or not X and Y are independent.

   Solution. (a)

		x = 0	x = 1
y	0	0.5	0.25
	1	0	0.25

   (b) The marginal distribution $g(x)$ for X is given by

$$g(0) = 0.5 + 0 = 0.5,$$
$$g(1) = 0.25 + 0.25 = 0.5,$$

   The marginal distribution $h(y)$ for Y is given by

$$h(0) = 0.5 + 0.25 = 0.75,$$
$$h(1) = 0 + 0.25 = 0.25.$$

   (c) They are not independent, for if $f(x, y)$ is the joint distribution, then for example

$$f(0, 0) = 0.5 \neq (0.5)(0.75) = g(0) \cdot h(0).$$

□

3. Let X and Y be discrete random variables with joint probability distribution given by the following table:

		x = 2	x = 3	x = 4
y	1	0.06	0.15	0.09
	2	0.14	0.35	0.21

   (a) Find the marginal distributions for X and Y.
   (b) Find the conditional distribution for X given $Y = 2$.
   (c) Determine whether or not X and Y are independent.

Solution. (a) The marginal distribution $g(x)$ for X is given by

$$g(2) = 0.06 + 0.14 = 0.2,$$
$$g(3) = 0.15 + 0.35 = 0.5,$$
$$g(4) = 0.09 + 0.21 = 0.3.$$

The marginal distribution $h(y)$ for Y is given by

$$h(1) = 0.06 + 0.15 + 0.09 = 0.3,$$
$$h(2) = 0.14 + 0.35 + 0.21 = 0.7.$$

(b) If we let $f(x, y)$ denote the joint distribution of X and Y, then the conditional distribution for X given $Y = 2$ is defined as,

$$f(x|2) = \frac{f(x,2)}{h(2)} = \frac{f(x,2)}{0.7}.$$

So we have

$$f(2|2) = \frac{0.14}{0.7} = 0.2,$$
$$f(3|2) = \frac{0.35}{0.7} = 0.5,$$
$$f(4|2) = \frac{0.21}{0.7} = 0.3.$$

(c) Recall that X and Y are independent if $f(x, y) = g(x) \cdot h(y)$ for all x, y. We need to check six cases:

$$f(2,1) = 0.06 = (0.2)(0.3) = g(2)h(1), \quad f(2,2) = 0.14 = (0.2)(0.7) = g(2)h(2),$$
$$f(3,1) = 0.15 = (0.5)(0.3) = g(3)h(1), \quad f(3,2) = 0.35 = (0.5)(0.7) = g(3)h(2),$$
$$f(4,1) = 0.09 = (0.3)(0.3) = g(4)h(1), \quad f(4,2) = 0.21 = (0.3)(0.7) = g(4)h(2).$$

Thus X and Y are independent.

□

4. Let X be a random variable with the following distribution

x	-2	-1	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Let $Y = X^2$.

(a) Find the distribution $g(y)$ of Y.

(b) Find the joint distribution $f(x, y)$ of X and Y.

(c) Find the marginal distributions of X and Y.

(d) Determine whether or not X and Y are independent.

Solution. (a) Since $Y = X^2$, the range of Y is $\{1, 4\}$, and

$$P(Y = 1) = P(X = -1) + P(X = 1), \quad P(Y = 4) = P(X = -2) + P(X = 2).$$

In summary, the distribution for Y is

y	1	4
$P(Y = y)$	$\frac{1}{2}$	$\frac{1}{2}$

(b) The joint distribution is

		x = -2	-1	1	2
y	1	0	$\frac{1}{4}$	$\frac{1}{4}$	0
	4	$\frac{1}{4}$	0	0	$\frac{1}{4}$

(c) Marginal distribution for X, $g(x) = \sum_y f(x,y)$:

$$g(-2) = \frac{1}{4}, \quad g(-1) = \frac{1}{4}, \quad g(1) = \frac{1}{4}, \quad g(2) = \frac{1}{4}$$

Marginal distribution for Y, $h(y) = \sum_x f(x,y)$:

$$h(1) = \frac{1}{2}, \quad h(4) = \frac{1}{2}$$

(d) Note that $f(-2,1) = 0 \neq g(-2) \cdot h(1) = \frac{1}{8}$. Therefore X and Y are not independent.

□

5. The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} x+y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the marginal densities for X and Y, and determine whether X and Y are independent.

Solution. The marginal density for X is

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x,y)\, dy \\ &= \int_0^1 x+y\, dy \\ &= xy + \frac{y^2}{2}\Big|_0^1 \\ &= x + \frac{1}{2} \end{aligned}$$

The marginal density for Y is

$$\begin{aligned} h(y) &= \int_{-\infty}^{\infty} f(x,y)\, dx \\ &= \int_0^1 x+y\, dx \\ &= \frac{x^2}{2} + xy\Big|_0^1 \\ &= \frac{1}{2} + y \end{aligned}$$

Then

$$g(x)h(y) = \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{4} \neq x + y = f(x,y)$$

(for example $g\left(\frac{3}{4}\right)h\left(\frac{3}{4}\right) = \frac{25}{16} \neq \frac{3}{2} = f\left(\frac{3}{4}, \frac{3}{4}\right)$). Therefore the random variables are not independent. □

6. Find the marginal densities of X and Y given their joint probability density

$$f(x,y) = \begin{cases} \frac{2}{5}(x+4y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Solution. The marginal density of X is

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\ dy = \int_{-\infty}^{0} f(x,y)\ dy + \int_{0}^{1} f(x,y)\ dy + \int_{1}^{\infty} f(x,y)\ dy$$

$$= \int_{-\infty}^{0} 0\ dy + \int_{0}^{1} \frac{2}{5}(x+4y)\ dy + \int_{1}^{\infty} 0\ dy = \left(\frac{2xy}{5} + \frac{4y^2}{5}\right)\Big|_0^1 = \frac{2x}{5} + \frac{4}{5}.$$

The marginal density of Y is

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\ dx = \int_{-\infty}^{0} f(x,y)\ dx + \int_{0}^{1} f(x,y)\ dx + \int_{1}^{\infty} f(x,y)\ dx$$

$$= \int_{-\infty}^{0} 0\ dx + \int_{0}^{1} \frac{2}{5}(x+4y)\ dx + \int_{1}^{\infty} 0\ dx = \left(\frac{x^2}{5} + \frac{8xy}{5}\right)\Big|_0^1 = \frac{1}{5} + \frac{8y}{5}.$$

□

7. Let X and Y be jointly continuous random variables with joint probability density given by

$$f(x,y) = \begin{cases} \frac{12}{5}(2x - x^2 - xy) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal densities for X and Y.

(b) Find the conditional density for X given $Y = y$ and the conditional density for Y given $X = x$.

(c) Compute the probability $P(\frac{1}{2} < X < 1 | Y = \frac{1}{4})$.

(d) Determine whether or not X and Y are independent.

Solution. (a) The marginal density $g(x)$ for X is

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x,y)\ dy \\ &= \int_{0}^{1} \frac{12}{5}(2x - x^2 - xy)\ dy \\ &= \frac{12}{5}\left(2xy - x^2y - \frac{xy^2}{2}\right)\Big|_0^1 \\ &= \frac{12}{5}\left(2x - x^2 - \frac{x}{2}\right) \\ &= \frac{18x}{5} - \frac{12x^2}{5}, \end{aligned}$$

for $0 < x < 1$ and 0 elsewhere.

The marginal density $h(y)$ for Y is

$$
\begin{aligned}
h(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \\
&= \int_0^1 \frac{12}{5}(2x - x^2 - xy)\,dx \\
&= \frac{12}{5}\left(x^2 - \frac{x^3}{3} - \frac{x^2 y}{2}\right)\Big|_0^1 \\
&= \frac{12}{5}\left(1 - \frac{1}{3} - \frac{y}{2}\right) \\
&= \frac{8}{5} - \frac{6y}{5},
\end{aligned}
$$

for $0 < y < 1$ and 0 elsewhere.

(b) The conditional density for X given $Y = y$ when $0 < x < 1, 0 < y < 1$ is given by,

$$
f(x|y) = \frac{f(x,y)}{h(y)} = \frac{\frac{12}{5}(2x - x^2 - xy)}{\frac{8}{5} - \frac{6y}{5}} = \frac{12x - 6x^2 - 6xy}{4 - 3y},
$$

and $f(x|y) = 0$ elsewhere.

The conditional density for Y given $X = x$ when $0 < x < 1, 0 < y < 1$ is given by,

$$
f(y|x) = \frac{f(x,y)}{g(x)} = \frac{\frac{12}{5}(2x - x^2 - xy)}{\frac{18x}{5} - \frac{12x^2}{5}} = \frac{4 - 2x - 2y}{3 - 2x},
$$

and $f(x|y) = 0$ elsewhere.

(c)

$$
\begin{aligned}
P\left(\frac{1}{2} < X < 1 \,\middle|\, Y = \frac{1}{4}\right) &= \int_{\frac{1}{2}}^{1} f\left(x \,\middle|\, \frac{1}{4}\right)\,dx \\
&= \int_{\frac{1}{2}}^{1} \frac{12x - 6x^2 - 6x(\frac{1}{4})}{4 - 3(\frac{1}{4})}\,dx \\
&= \int_{\frac{1}{2}}^{1} \frac{42x}{13} - \frac{24x^2}{13}\,dx \\
&= \frac{21x^2}{13} - \frac{8x^3}{13}\Big|_{\frac{1}{2}}^{1} \\
&= \left(\frac{21(1)^2}{13} - \frac{8(1)^3}{13}\right) - \left(\frac{21(\frac{1}{2})^2}{13} - \frac{8(\frac{1}{2})^3}{13}\right) \\
&= \frac{35}{52} \\
&\approx 0.6731
\end{aligned}
$$

(d) They are not independent. For example

$$
f(0.25, 0.25) = 0.9 \neq 0.975 = g(0.25) \cdot h(0.25).
$$

□

8. Let X and Y be discrete random variables with joint probability distribution given by the following table:

		x		
		-3	2	4
y	1	0.1	0.2	0.2
	3	0.3	0.1	0.1

(a) Find the conditional distribution for X given $Y = 1$.

(b) Are X and Y independent? Justify your answer.

Solution. (a) If we let $f(x, y)$ denote the joint distribution of X and Y, then the conditional distribution for X given $Y = 1$ is defined as,

$$f(x|1) = \frac{f(x, 1)}{h(1)} = \frac{f(x, 1)}{0.5}.$$

So we have

$$f(-3|1) = \frac{0.1}{0.5} = 0.2, \quad f(2|1) = \frac{0.2}{0.5} = 0.4, \quad f(4|1) = \frac{0.2}{0.5} = 0.4.$$

(b) No, X and Y are not independent. For example

$$f(-3, 1) = 0.1 \neq (0.4)(0.5) = g(-3) \cdot h(1)$$

□

9. Given the joint probability density

$$f(x, y) = \begin{cases} \frac{2}{3}(x + 2y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the conditional distribution of X given $Y = y$ and use it to evaluate $P(X \leq \frac{1}{2}|Y = \frac{1}{2})$.

Solution. The definition for conditional distribution of X given $Y = y$ is

$$f(x|y) = \frac{f(x, y)}{h(y)}$$

where $h(y)$ is the marginal distribution for Y. Then

$$\begin{aligned} h(y) &= \int_{-\infty}^{\infty} f(x, y)\, dx \\ &= \int_0^1 \frac{2}{3}(x + 2y)\, dx \\ &= \frac{x^2}{3} + \frac{4}{3}xy \Big|_0^1 \\ &= \frac{1}{3}(1 + 4y) \end{aligned}$$

So for $0 < x < 1$ we have

$$f(x|y) = \frac{\frac{2}{3}(x + 2y)}{\frac{1}{3}(1 + 4y)} = \frac{2x + 4y}{1 + 4y}$$

and $f(x|y) = 0$ elsewhere. In particular

$$f\left(x \middle| \frac{1}{2}\right) = \frac{2x + 4(\frac{1}{2})}{1 + 4(\frac{1}{2})} = \frac{2x+2}{3}.$$

Thus

$$\begin{aligned} P\left(X \le \frac{1}{2} \middle| Y = \frac{1}{2}\right) &= \int_0^{\frac{1}{2}} \frac{2x+2}{3}\, dx \\ &= \frac{x^2}{3} + \frac{2x}{3}\Big|_0^{\frac{1}{2}} \\ &= \frac{5}{12}. \end{aligned}$$

□

10. The joint probability density function for continuous random variables is given below. Let $f(x|y)$ be the conditional density for X given $Y = y$. Find $P(0 \le X \le \frac{1}{2}|Y = 1)$.

$$f(x,y) = \begin{cases} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution. The marginal density for Y is

$$\begin{aligned} h(y) &= \int_{-\infty}^{\infty} f(x,y)\, dx \\ &= \int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\, dx \\ &= \frac{6}{7}\left(\frac{x^3}{3} + \frac{x^2 y}{4}\right)\Big|_0^1 \\ &= \frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right) \end{aligned}$$

so the conditional density is

$$f(x|y) = \frac{f(x,y)}{g(y)} = \frac{\frac{6}{7}\left(x^2 + \frac{xy}{2}\right)}{\frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right)} = \frac{12x^2 + 6xy}{4 + 3y}.$$

Thus

$$\begin{aligned} P\left(0 \le X \le \frac{1}{2} \middle| Y = 1\right) &= \int_0^{\frac{1}{2}} f(x|1)\, dx \\ &= \int_0^{\frac{1}{2}} \frac{12x^2 + 6x}{7}\, dx \\ &= \frac{4x^3}{7} + \frac{3x^2}{7}\Big|_0^{\frac{1}{2}} \\ &= \frac{5}{28}. \end{aligned}$$

□